

BAIRE REDUCTIONS AND GOOD BOREL REDUCIBILITIES

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ABSTRACT. In [8] we have considered a wide class of “well-behaved” reducibilities for sets of reals. In this paper we continue with the study of Borel reducibilities by proving a dichotomy theorem for the degree-structures induced by good Borel reducibilities. This extends and improves the results of [8] allowing to deal with a larger class of notions of reduction (including, among others, the Baire class ξ functions).

1. INTRODUCTION

A *reducibility* for sets of reals¹ is simply a collection \mathcal{F} of functions from \mathbb{R} to \mathbb{R} which is used to reduce a set of reals to another one: given $A, B \subseteq \mathbb{R}$, we say that A is \mathcal{F} -reducible to B just in case there is some $f \in \mathcal{F}$ such that $x \in A \iff f(x) \in B$ for every $x \in \mathbb{R}$. Such an \mathcal{F} allows to measure the “relative complexity” of the sets of reals, and \mathcal{F} itself can be viewed as the “unit of measurement” that we are using: in general, the “smaller” is our set \mathcal{F} , the more accurate is our measurement (i.e. the finer is the hierarchy of degrees induced by \mathcal{F}).

The first two reducibilities that one encounters in the literature are the collection W of all continuous functions and the collection L of all Lipschitz functions with constant less than or equal to 1. The corresponding degree-structures were extensively studied (assuming AD, the *Axiom of Determinacy*) by Wadge, Steel, Van Wesep and many other set theorists (Martin, Kechris, Louveau, Saint-Raymond to name a few), and have had many applications in Set Theory and Theoretical Computer Science (see for example [11] or [5]). Some years ago, Andretta and Martin considered the collection Bor of all Borel functions and the collection D_2 of all Δ_2^0 -functions, and they proved that in both cases the degree-structures induced by those reducibilities look like the Wadge one, i.e. like the one induced by continuous functions. In [8] we have described a general method to extend this analysis to the so-called *Borel-amenable* reducibilities, among which there are e.g. the continuous functions, the Borel functions, the collection D_ξ of all Δ_ξ^0 -functions for $\xi < \omega_1$, i.e. the collection of those f such that $f^{-1}(D) \in \Delta_\xi^0$ for every $D \in \Delta_\xi^0$, and so on. As for the previous cases, we have obtained that whenever \mathcal{F} is a Borel-amenable set of reductions the degree-structure induced by \mathcal{F} looks like the Wadge one.

Since the D_ξ ’s form a natural stratification of the Borel functions, the present work was mainly motivated by the natural idea of considering the other classical stratification of the Borel functions, namely the *Baire class* ξ functions. Note that although in [7] it has been pointed out that there is a link between the two stratifications, from the point of view of reducibilities between sets of reals they clearly have a very different behaviour: in fact, we will prove that the Baire functions, contrarily to the case of the D_ξ ’s, induce a degree-structure which looks like the structure of

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¹As usual, we will always identify \mathbb{R} with the Baire space ${}^\omega\omega$ and call its elements “reals”.

the \mathbf{L} -degrees. This result is obtained comparing again the Baire stratification with the Delta stratification, and showing that the first one gives reducibilities which are *equivalent* to (i.e. induce the same degree-structures as) the ones obtained glueing together *chains* of Borel-amenable sets of reductions. On the way of studying these classes of functions, we will introduce the notion of *good Borel reducibility* which considerably extends the definition of Borel-amenability given in [8] (in fact it includes, among others, all the examples quoted in this introduction): building on our previous results, we will give a new general method to study these reducibilities, which will lead to the following dichotomy theorem.

Theorem 1.1. *Assume $\mathbf{AD} + \mathbf{DC}(\mathbb{R})$. If \mathcal{F} is a good Borel set of reductions then it induces either a Lipschitz-like or a Wadge-like hierarchy of degrees.*

This improves many of the results obtained in [8] and is a first step toward proving the naïve conjecture that the dichotomy above should hold for all “reasonable” Borel sets of reductions.

The paper is organized as follows: in Section 2 we will fix some notation and review some of the results about \mathbf{L} -degrees and Borel-amenable reducibilities that will be needed for the rest of the work. (We will systematically omit the proofs of these results — the reader interested in some of them can consult [13], [12] or the more succinct [3] for Lipschitz degrees, and [8] for Borel-amenable sets of reductions.) In Section 3 we will give the definition of good Borel reducibility, while in Section 4 we will introduce the Strong Decomposition Property and prove some results which essentially form the framework of the proof of our dichotomy theorem. In Section 5 we will deal with the cases of Lipschitz functions and chains of reductions, and these results will in turn be used in Section 6 to analyze the hierarchies of degrees induced by Baire functions. Finally, in Section 7 we will show how to compare different degrees-structures (in particular showing how to obtain a certain hierarchy from the finer ones).

2. PRELIMINARIES

Unless otherwise stated, we will always assume $\mathbf{ZF} + \mathbf{SLO}^L + \neg\mathbf{FS} + \mathbf{DC}(\mathbb{R})$ (see [8] and [3] for the definitions and for a brief account on these axioms). Anyway \mathbf{SLO}^L and $\neg\mathbf{FS}$ are easy consequences of \mathbf{AD} , thus one can also safely work in the most well-known theory $\mathbf{ZF} + \mathbf{AD} + \mathbf{DC}(\mathbb{R})$. In both cases, all the “determinacy axioms” are used in a local way throughout the paper, thus e.g. to compare Borel sets is enough to assume Borel-determinacy. Our notation and terminology is quite standard, and we systematically refer the reader to [8] for the basic definitions. We just recall here that ${}^A B$ denotes the set of all functions from A into B , that a set $F \subseteq {}^{\omega} 2$ is said *flip-set* whenever $\exists! n (z(n) \neq w(n)) \Rightarrow (z \in F \iff w \notin F)$ for every $z, w \in {}^{\omega} 2$, and that, given any pointclass $\Gamma \subseteq \mathcal{P}(\mathbb{R})$, a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is said Γ -*function* if $f^{-1}(D) \in \Gamma$ for every $D \in \Gamma$.

Let now $\mathcal{F} \subseteq {}^{\mathbb{R}} \mathbb{R}$ be a family of functions which is closed under composition, contains \mathbf{L} and admits a surjection $j: \mathbb{R} \rightarrow \mathcal{F}$, that is a so-called *set of reductions*. Recall that $A \leq_{\mathcal{F}} B \iff A = f^{-1}(B)$ for some $f \in \mathcal{F}$ (notice that $A \leq_{\mathcal{F}} B \iff \neg A \leq_{\mathcal{F}} \neg B$), and let $<_{\mathcal{F}}$ be the strict relation associated to $\leq_{\mathcal{F}}$. Since $\leq_{\mathcal{F}}$ is a preorder, we can canonically define the equivalence relation $\equiv_{\mathcal{F}}$ and study the partial order \leq induced by $\leq_{\mathcal{F}}$ on the equivalence classes of $\equiv_{\mathcal{F}}$, which are called \mathcal{F} -*degrees*. A set A (or its \mathcal{F} -degree $[A]_{\mathcal{F}}$) is said to be \mathcal{F} -*selfdual* if and only if $A \leq_{\mathcal{F}} \neg A$ (otherwise it is \mathcal{F} -*nonselfdual*), and $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ is called *nonselfdual pair*

whenever $A \not\leq_{\mathcal{F}} \neg A$. The *Semi-Linear Ordering Principle* for \mathcal{F} is the statement

$$(\text{SLO}^{\mathcal{F}}) \quad \forall A, B \subseteq \mathbb{R} (A \leq_{\mathcal{F}} B \vee \neg B \leq_{\mathcal{F}} A).$$

Under $\text{SLO}^{\mathcal{F}}$ we have that if A and B are $\leq_{\mathcal{F}}$ -incomparable, then $B \equiv_{\mathcal{F}} \neg A$: thus the ordering induced on the \mathcal{F} -degrees is *almost* a linear-order (it becomes indeed linear if each degree is identified with its dual). If now $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R} \mathbb{R}$ are sets of reductions, then $\leq_{\mathcal{G}}$ is clearly coarser than $\leq_{\mathcal{F}}$: hence $A \leq_{\mathcal{F}} B \Rightarrow A \leq_{\mathcal{G}} B$, if A is \mathcal{F} -selfdual then it is also \mathcal{G} -selfdual, and $[A]_{\mathcal{F}} \subseteq [A]_{\mathcal{G}}$. Moreover the following basic lemma holds.

Lemma 2.1 (ZF). *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R} \mathbb{R}$ be two sets of reductions. Then $\text{SLO}^{\mathcal{F}} \Rightarrow \text{SLO}^{\mathcal{G}}$, and assuming $\text{SLO}^{\mathcal{F}}$ we have $\forall A, B \subseteq \mathbb{R} (A <_{\mathcal{G}} B \Rightarrow A <_{\mathcal{F}} B)$.*

This lemma will be mostly used when $\mathcal{F} = \mathbb{L}$: this in particular means that under our axiomatization we also have $\text{SLO}^{\mathcal{F}}$ for *every* set of reductions \mathcal{F} . Moreover it easily implies that $\leq_{\mathcal{F}}$ is well-founded (since $\leq_{\mathbb{L}}$ is): therefore we can associate a rank $\|\cdot\|_{\mathcal{F}}$ to each set $A \subseteq \mathbb{R}$ (resp. \mathcal{F} -degree $[A]_{\mathcal{F}}$), and speak of *successor* and *limit* sets (resp. \mathcal{F} -degrees), and of the *cofinality* of a set (resp. of an \mathcal{F} -degree) with the obvious meaning. The next theorem sum up the general properties of sets of reductions — see Theorem 3.1 in [8]. Recall that given $A, B, A_n \in \mathbb{R}$, $\bigoplus_n A_n$ denotes the set $\bigcup_n (n^\frown A_n)$, while $A \oplus B$ denotes $\bigoplus_n C_n$, where $C_{2k} = A$ and $C_{2k+1} = B$ for every $k \in \omega$.

Theorem 2.2. *Let $\mathcal{F} \subseteq \mathbb{R} \mathbb{R}$ be a set of reductions. Then*

- i) $\text{lh}(\leq_{\mathcal{F}}) = \Theta$, where $\Theta = \sup\{\alpha \mid f: \mathbb{R} \twoheadrightarrow \alpha \text{ for some surjection } f\}$;
- ii) anti-chains have size at most 2 and are of the form $\{[A]_{\mathcal{F}}, [\neg A]_{\mathcal{F}}\}$ for some set A ;
- iii) $\mathbb{R} \not\leq_{\mathcal{F}} \neg \mathbb{R} = \emptyset$ and if $A \neq \emptyset, \mathbb{R}$ then $\emptyset, \mathbb{R} <_{\mathcal{F}} A$;
- iv) if $A \not\leq_{\mathcal{F}} \neg A$ then $A \oplus \neg A$ is \mathcal{F} -selfdual and is the successor of both A and $\neg A$.
In particular, after an \mathcal{F} -nonselfdual pair there is a single \mathcal{F} -selfdual degree;
- v) if $A_0 <_{\mathcal{F}} A_1 <_{\mathcal{F}} \dots$ is a countable \mathcal{F} -chain of subsets of \mathbb{R} then $\bigoplus_n A_n$ is \mathcal{F} -selfdual and is the supremum of these sets. In particular if $[A]_{\mathcal{F}}$ is limit of countable cofinality then $A \leq_{\mathcal{F}} \neg A$;
- vi) if $A \not\leq_{\mathcal{F}} \neg A$ and $\mathcal{G} \subseteq \mathcal{F}$ is another set of reductions then $[A]_{\mathcal{F}} = [A]_{\mathcal{G}}$. In particular, $[A]_{\mathcal{F}} = [A]_{\mathbb{L}}$.

Thus to determine the hierarchy of degrees induced by some \mathcal{F} we have only to understand what happens after a single selfdual degree and at limit levels of uncountable cofinality.

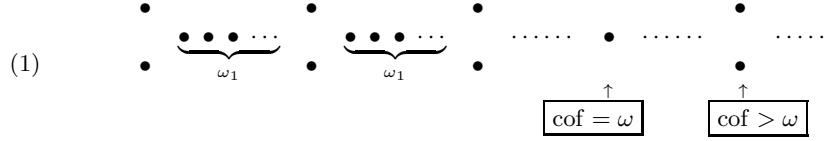
Given any set of reductions \mathcal{F} (or even just any *set of functions*) we can define its *characteristic set* $\Delta_{\mathcal{F}} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathcal{F}} \mathbf{N}_{(0)}\}$, which is formed by all sets $A \subseteq \mathbb{R}$ which are *simple* from the “point of view” of \mathcal{F} . As a simple excercise one can check that $\Delta_{\mathbb{D}_{\xi}} = \Delta_{\xi}^0$ for every countable ξ , and that $\Delta_{\text{Bor}} = \Delta_1^1$. It is easy to see that if \mathcal{F} is closed under composition then every $f \in \mathcal{F}$ is a $\Delta_{\mathcal{F}}$ -function (even if the converse is not always true — see [8] for a counter-example), thus it make sense to define the *saturation* of \mathcal{F}

$$\text{Sat}(\mathcal{F}) = \{f \in \mathbb{R} \mathbb{R} \mid f \text{ is a } \Delta_{\mathcal{F}}\text{-function}\},$$

and to say that \mathcal{F} is *saturated* just in case $\mathcal{F} = \text{Sat}(\mathcal{F})$. Moreover it is easy to see that $\mathcal{F} \subseteq \mathcal{G}$ implies $\Delta_{\mathcal{F}} \subseteq \Delta_{\mathcal{G}}$ (the converse is not true in general, unless $\mathcal{G} = \text{Bor}$ or $\mathcal{G} = \mathbb{D}_{\xi}$ for some $\xi < \omega_1$, and $\mathbf{N}_s \in \Delta_{\mathcal{F}}$ for every $s \in {}^{<\omega} \omega$). Finally, \mathcal{F} is said to be *Borel* if $\{\mathbf{N}_s \mid s \in {}^{<\omega} \omega\} \subseteq \Delta_{\mathcal{F}} \subseteq \Delta_1^1$, that is if $\mathcal{F} \subseteq \text{Bor}$ and \mathcal{F} recognizes as simple all the basic clopen sets of \mathbb{R} .

Here are some basic facts about \mathbb{L} -degrees: after a selfdual \mathbb{L} -degree there is always another selfdual \mathbb{L} -degree, and a limit \mathbb{L} -degree is selfdual if and only if it is

of countable cofinality, otherwise it is nonselfdual. Thus after a selfdual \mathbb{L} -degree $[A]_{\mathbb{L}}$ there is always an ω_1 -chain of consecutive selfdual \mathbb{L} -degrees, and therefore the \mathbb{L} -hierarchy looks like this:



Any structure of this kind will be called *Lipschitz-like*.

Now we turn our attention to Borel-amenable sets of reductions.

Definition 1. A set of reductions \mathcal{F} is *Borel-amenable* if:

- i) $\text{Lip} \subseteq \mathcal{F} \subseteq \text{Bor}$;
- ii) for every $\Delta_{\mathcal{F}}$ -partition $\langle D_n \mid n \in \omega \rangle$ and every collection $\{f_n \mid n \in \omega\} \subseteq \mathcal{F}$ we have that

$$f = \bigcup_{n \in \omega} (f_n \upharpoonright D_n) \in \mathcal{F},$$

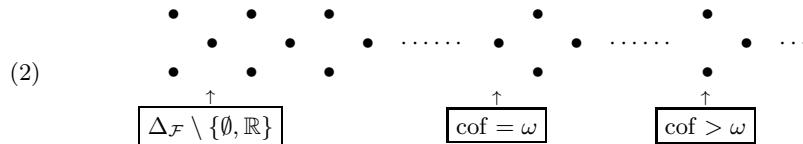
where Lip is the collection of *all* Lipschitz functions (with any constant).

As an example of Borel-amenable reducibility one can take any of the D_{ξ} 's or Bor . It turns out that for every Borel-amenable set of reductions \mathcal{F} there is some $\xi \leq \omega_1$ (called the *level* of \mathcal{F}) such that $\Delta_{\mathcal{F}} = \Delta_{\xi}^0$ (where, with a little abuse of notation, we put $\Delta_{\omega_1}^0 = \Delta_1^1$): thus, in particular, $\text{Sat}(\mathcal{F})$ is always either one of the D_{ξ} 's or Bor . Moreover for any Borel-amenable set of reductions \mathcal{F} we have the following lemma.

Lemma 2.3. Let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$ -partition of \mathbb{R} and let $A \neq \mathbb{R}$.

- a) $\forall n \in \omega (A \cap D_n \leq_{\mathcal{F}} A)$.
- b) If $C \subseteq \mathbb{R}$ and $A \cap D_n \leq_{\mathcal{F}} C$ for every $n \in \omega$ then $A \leq_{\mathcal{F}} C$.
- c) If $\forall n \in \omega (A \cap D_n <_{\mathcal{F}} A)$ then $A \leq_{\mathcal{F}} \neg A$. Moreover, if $D_n = \emptyset$ for all but finitely many n 's then A is not limit.

Let us say that \mathcal{F} has the *Decomposition Property* (**DP** for short) if for every selfdual $A \notin \Delta_{\mathcal{F}}$ there is a $\Delta_{\mathcal{F}}$ -partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} such that $A \cap D_n <_{\mathcal{F}} A$ for every n or, equivalently, if for every selfdual A which is \mathbb{L} -minimal in $[A]_{\mathcal{F}}$ one has that $A \leq_{\mathbb{L}} \neg A$ (and A is either limit or successor of a nonselfdual pair with respect to $\leq_{\mathbb{L}}$). Using this property (which turns out to be a consequence of Borel-amenability) we have proved in [8] that both after a single selfdual degree and at limit levels of uncountable cofinality there is a nonselfdual pair. Thus the hierarchy of degrees induced by \mathcal{F} looks like this:



Any structure of this kind will be called *Wadge-like*. Notice that one gets both the Wadge hierarchy and the degree-structures induced by Bor and D_2 as particular instances of the previous result.

The **DP** allows also to compare different sets of reductions by means of the degree-structures induced by them. Let us say that two sets of reductions \mathcal{F} and \mathcal{G} are *equivalent* ($\mathcal{F} \simeq \mathcal{G}$ in symbols) if they induce the same hierarchy of degrees, that is if for every $A, B \subseteq \mathbb{R}$ we have $A \leq_{\mathcal{F}} B \iff A \leq_{\mathcal{G}} B$: if \mathcal{F} and \mathcal{G} are Borel-amenable sets of reductions then

$$\mathcal{F} \simeq \mathcal{G} \iff \Delta_{\mathcal{F}} = \Delta_{\mathcal{G}},$$

that is $\mathcal{F} \simeq \mathcal{G}$ if and only if their degree-structures coincide on the first nontrivial level. In particular, since $\Delta_{\mathcal{F}} = \Delta_{\text{Sat}(\mathcal{F})}$, we have that $\mathcal{F} \simeq \text{Sat}(\mathcal{F})$.

We want to conclude this section by recalling how to construct, given an \mathcal{F} -selfdual set $A \subseteq \mathbb{R}$, where \mathcal{F} is of level $\xi < \omega_1$, its successor degree(s). Fix an increasing sequence of ordinals $\langle \mu_n \mid n \in \omega \rangle$ cofinal in ξ , and a sequence of sets P_n such that $P_n \in \Pi_{\mu_n}^0 \setminus \Sigma_{\mu_n}^0$. Let $\langle \cdot, \cdot \rangle: \omega \times \omega \rightarrow \omega$ be any bijection, and for any zero-dimensional space² \mathcal{X} define the homeomorphism

$$\bigotimes^{\mathcal{X}}: {}^{\omega}({}^{\omega}\mathcal{X}) \rightarrow {}^{\omega}\mathcal{X}: \langle x_n \mid n \in \omega \rangle \mapsto x = \bigotimes_n^{\mathcal{X}} x_n,$$

by letting $x(\langle n, m \rangle) = x_n(m)$, and, conversely, the “projections” $\pi_n^{\mathcal{X}}: {}^{\omega}\mathcal{X} \rightarrow {}^{\omega}\mathcal{X}$ by setting $\pi_n^{\mathcal{X}}(x) = \langle x(\langle n, m \rangle) \mid m \in \omega \rangle$ (clearly, every “projection” is surjective, continuous and open). Note that given a sequence of functions $f_n: {}^{\omega}\mathcal{X} \rightarrow {}^{\omega}\mathcal{X}$, we can use the homeomorphism $\bigotimes^{\mathcal{X}}$ to define the function

$$\bigotimes^{\mathcal{X}} \langle f_n \mid n \in \omega \rangle = \bigotimes_n^{\mathcal{X}} f_n: {}^{\omega}\mathcal{X} \rightarrow {}^{\omega}\mathcal{X}: x \mapsto \bigotimes_n^{\mathcal{X}} f_n(x),$$

and it is not hard to check that $\bigotimes_n^{\mathcal{X}} f_n$ is continuous if and only if all the f_n ’s are continuous. Now consider the sets

$$\Sigma^{\xi}(A) = \{x \in \mathbb{R} \mid \exists n(\pi_{2n}(x) \in P_n \wedge \forall i < n(\pi_{2i}(x) \notin P_i) \wedge \pi_{2n+1}(x) \in A)\}$$

and

$$\Pi^{\xi}(A) = \Sigma^{\xi}(A) \cup R_{\xi},$$

where $R_{\xi} = \{x \in \mathbb{R} \mid \forall n(\pi_{2n}(x) \notin P_n)\}$: it turns out that for every $A \leq_{\mathcal{F}} \neg A$ the sets $\Sigma^{\xi}(A)$ and $\Pi^{\xi}(A)$ are $\leq_{\mathcal{F}}$ -incomparable and are the immediate successors of A .

3. GOOD BOREL REDUCIBILITIES

In [8] we have studied a special kind of Borel reducibilities but, as we will see later in this paper, there are also other “natural” sets of reductions which are not of this kind (namely Lipschitz functions, uniformly continuous functions, Baire functions, and so on). Thus our goal is to weaken the condition of Borel-amenability in order to be able to study also these other examples. Recall that the second condition of Borel-amenability says that $f = \bigcup_n (f_n \upharpoonright D_n) \in \mathcal{F}$ whenever $\{f_n \mid n \in \omega\} \subseteq \mathcal{F}$ and $\langle D_n \mid n \in \omega \rangle$ is a $\Delta_{\mathcal{F}}$ -partition of \mathbb{R} . We will weaken this condition both allowing to use only f_n ’s which are in \mathbb{L} and using the concept of *boundness* (in a pointclass): a pointclass Λ is (\mathbb{L}) -*bounded* in an \mathbb{L} -pointclass Γ if there is some $A \in \Gamma$ such that $B \leq_{\mathbb{L}} A$ for every $B \in \Lambda$ (which in particular implies that $\Lambda \subseteq \Gamma$). Moreover, $\langle D_n \mid n \in \omega \rangle$ is a Γ -*bounded partition* of \mathbb{R} if it is a Γ -partition of \mathbb{R} such that $\{D_n \mid n \in \omega\}$ is bounded in Γ .

Definition 2. We say that \mathcal{F} satisfies the *partitioning condition* (**PC** for short) if for every $\Delta_{\mathcal{F}}$ -*bounded* partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} and every collection $\{f_n \mid n \in \omega\} \subseteq \mathbb{L}$ one has that

$$f = \bigcup_{n \in \omega} (f_n \upharpoonright D_n) \in \mathcal{F}.$$

Notice that there are just three types of Borel reducibilities that can satisfy the **PC**:

TYPE I: $\Delta_{\mathcal{F}} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathbb{L}} \mathbf{N}_s \text{ for some } s \in {}^{<\omega}\omega\}$;

TYPE II: $\Delta_{\mathcal{F}} = \Delta_{\xi}^0$ for some countable ξ or $\Delta_{\mathcal{F}} = \Delta_1^1$;

TYPE III: $\Delta_{\mathcal{F}} = \Delta_{<\lambda}^0 = \bigcup_{\mu < \lambda} \Delta_{\mu}^0$ for some countable limit ordinal λ .

²When $\mathcal{X} = \omega$ we will simply drop the symbol \mathcal{X} in all the relevant notation.

The proof of this fact is essentially the same of Proposition 4.3 in [8]. By Borel determinacy we need to consider just two cases³, namely $\Delta_1^0 \subsetneq \Delta_{\mathcal{F}}$ and $\Delta_{\mathcal{F}} \subseteq \Delta_1^0$. In the first case, assume that $\Delta_{\mathcal{F}} \neq \Delta_1^1$ and that \mathcal{F} is not of type III, and let $1 < \xi < \omega_1$ be the smallest ordinal such that $\Delta_{\mathcal{F}} \subseteq \Delta_\xi^0$. If $D \in \Delta_\xi^0$, then there is some partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} such that $D = \bigcup_{i \in I} D_i$ for some $I \subseteq \omega$ and $D_n \in \Pi_{\mu_n}^0$ for some $\mu_n < \xi$ (see Theorem 4.2 in [7]). Since $\Delta_\mu^0 \subseteq \Delta_{\mathcal{F}}$ for every $\mu < \xi$ (by minimality of ξ), we have that $\bigcup_{\mu < \xi} \Pi_\mu^0 \subseteq \Delta_{\mathcal{F}}$ by Borel determinacy again, and hence that $\{D_n \mid n \in \omega\} \subseteq \bigcup_{\mu < \xi} \Pi_\mu^0$ is bounded in $\Delta_{\mathcal{F}}$ (when ξ is limit use the fact that \mathcal{F} is not of type III). Let g_0, g_1 be the constant functions with value $\vec{0}$ and $\vec{1}$, respectively, and put $f_i = g_0$ if $i \in I$ and $f_i = g_1$ otherwise. By the **PC**, $f = \bigcup_{n \in \omega} (f_n \upharpoonright D_n) \in \mathcal{F}$ and $f^{-1}(\mathbf{N}_{\langle 0 \rangle}) = D$, i.e. $D \in \Delta_{\mathcal{F}}$: therefore $\Delta_\xi^0 \subseteq \Delta_{\mathcal{F}}$ and \mathcal{F} is of type II. Finally, the argument for the case $\Delta_{\mathcal{F}} \subseteq \Delta_1^0$ is similar to the previous one (it suffices to prove that if \mathcal{F} is not of type I then $\Delta_{\mathcal{F}} = \Delta_1^0$), and it is left to the reader.

As a corollary, one gets that $\Delta_{\mathcal{F}}$ is an algebra of sets (i.e. it is closed under complementation and finite intersections). Moreover, the **PC** implies $\mathcal{F} \supseteq \mathbf{L}$: therefore, since $\Delta_{\mathcal{F}} \subseteq \Delta_1^1$ already implies that there is a surjection $j: \mathbb{R} \twoheadrightarrow \mathcal{F}$, a Borel set of functions \mathcal{F} which satisfies the **PC** is also a *set of reductions* just in case it is closed under composition.

Another consequence of the **PC** is Lemma 4.4 of [8] (since any finite $\Delta_{\mathcal{F}}$ -partition of \mathbb{R} is obviously bounded in $\Delta_{\mathcal{F}}$): if $D \subseteq D'$ are in $\Delta_{\mathcal{F}}$ and $A \subseteq \mathbb{R}$ is such that $A \cap D' \neq \mathbb{R}$ then $A \cap D \leq_{\mathcal{F}} A \cap D'$ (in particular, if $A \neq \mathbb{R}$ then $A \cap D \leq_{\mathcal{F}} A$ for every $D \in \Delta_{\mathcal{F}}$). Finally, the **PC** allows to reprove Lemma 2.3 (using almost the same argument) in case \mathcal{F} is a Borel set of reductions which satisfies the **PC** (but non necessarily a Borel-amenable one) as soon as the partition $\langle D_n \mid n \in \omega \rangle$ is bounded in $\Delta_{\mathcal{F}}$ and part b) is replaced by the following condition:

$$(\star) \quad \text{if } C \subseteq \mathbb{R} \text{ and } A \cap D_n \leq_{\mathbf{L}} C \text{ for every } n \text{ then } A \leq_{\mathcal{F}} C.$$

Besides the **PC**, there is also another condition which is somewhat hidden in the definition of Borel-amenable.

Definition 3. If Δ is an \mathbf{L} -pointclass, we say that an arbitrary function $f: \mathbb{R} \rightarrow \mathbb{R}$ is *σ -bounded (in Δ)* if for every countable collection $\{D_n \mid n \in \omega\}$ bounded in Δ one has that $\{f^{-1}(D_n) \mid n \in \omega\}$ is bounded in Δ as well (thus, in particular, f is a Δ -function). A set of functions \mathcal{F} satisfies the *σ -boundness condition (σ -**BC** for short)* if every $f \in \mathcal{F}$ is σ -bounded in $\Delta_{\mathcal{F}}$.

It is easy to check that if \mathcal{F} is of type II then *every* countable collection $\{D_n \mid n \in \omega\} \subseteq \Delta_{\mathcal{F}}$ is bounded in $\Delta_{\mathcal{F}}$, thus **σ -BC** becomes relevant only when \mathcal{F} is not of type II (this is the reason for which this condition was not explicitly highlighted in [8]). On the other hand, if \mathcal{F} is of type I or III the **σ -BC** turns out to be equivalent to the seemingly stronger statement “if $f \in \mathcal{F}$ and Γ is bounded in $\Delta_{\mathcal{F}}$ (with Γ of arbitrary size) then $\{f^{-1}(C) \mid C \in \Gamma\}$ is bounded in $\Delta_{\mathcal{F}}$ ”: this is because in the cases under consideration $\Delta_{\mathcal{F}}$ has “countable cofinality” (i.e. there is a countable chain which is \mathbf{L} -unbounded in $\Delta_{\mathcal{F}}$), and therefore from every pointclass \mathbf{L} -unbounded in $\Delta_{\mathcal{F}}$ one can extract a countable subpointclass which is still \mathbf{L} -unbounded in $\Delta_{\mathcal{F}}$.

We will see that the **PC** and the **σ -BC** are strong enough to civilize the hierarchy of degrees induced by \mathcal{F} , so let us give the following definition:

³The principle **SLO^L** for Borel sets, which follows from Borel determinacy, implies that for every pair of \mathbf{L} -pointclasses $\Gamma, \Lambda \subseteq \Delta_1^1$ either $\Gamma \subseteq \Lambda$ or $\Lambda \subseteq \Gamma$: therefore if both Γ and Λ are selfdual either $\Gamma \subseteq \Lambda$ or $\Lambda \subseteq \Gamma$.

Definition 4. A Borel set of reductions is said to be a *good Borel reducibility* if it satisfies both the **PC** and the **σ -BC**. The collection of all good Borel reducibilities will be denoted by **GR**.

Note that the Borel-amenable reducibilities form a *proper* subset of good Borel reducibilities of type II, as $\text{Lip} \not\subseteq D_\xi^L$ for any $\xi < \omega_1$, where D_ξ^L is the collection of those f which are in L on a (countable) Δ_ξ^0 -partition (in particular this proves that each D_ξ^L does not contain *any* Borel-amenable set of reductions). In fact, one can easily check that the pseudoidentity $\text{id}^- : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto \langle x(n+1) \mid n \in \omega \rangle$ is such that for *every* countable partition $\langle D_n \mid n \in \omega \rangle$ and *every* family $\{f_n \mid n \in \omega\} \subseteq L$ there is some n_0 such that $f_{n_0} \upharpoonright D_{n_0} \neq \text{id}^- \upharpoonright D_{n_0}$ (the argument is based on the Baire Category Theorem and is almost identical to the one used in Remark 6.2 of [8]).

4. THE STRONG DECOMPOSITION PROPERTY AND THE DICHOTOMY THEOREM

First we want to prove that every good Borel reducibility \mathcal{F} has the following bounded version of the Decomposition Property.

Definition 5. A set $A \subseteq \mathbb{R}$ has the *Strong Decomposition Property* with respect to a Borel set of reductions \mathcal{F} if there is a $\Delta_{\mathcal{F}}$ -*bounded* partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} such that $A \cap D_n <_{\mathcal{F}} A$ for every n .

A Borel set of reductions has the *Strong Decomposition Property* (**SDP** for short) if every $A \subseteq \mathbb{R}$ such that $A \leq_{\mathcal{F}} \neg A$ and $A \notin \Delta_{\mathcal{F}}$ has the Strong Decomposition Property with respect to \mathcal{F} .

Observe that if \mathcal{F} is of type II then the **SDP** is equivalent to the **DP** by the observation following Definition 3.

Remark 4.1. If \mathcal{F} satisfies the **σ -BC**, then A has the Strong Decomposition Property with respect to \mathcal{F} if and only if there is some B in $[A]_{\mathcal{F}}$ which has the Strong Decomposition Property with respect to \mathcal{F} . In fact, let $f \in \mathcal{F}$ be a reduction of A into B , and let $\langle D'_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$ -bounded partition of \mathbb{R} such that $B \cap D'_n <_{\mathcal{F}} B$. Put $D_n = f^{-1}(D'_n)$: $\langle D_n \mid n \in \omega \rangle$ is a $\Delta_{\mathcal{F}}$ -bounded partition of \mathbb{R} by the **σ -BC**, and since f witnesses $A \cap D_n \leq_{\mathcal{F}} B \cap D'_n$ we have also $A \cap D_n \leq_{\mathcal{F}} B \cap D'_n <_{\mathcal{F}} B \leq_{\mathcal{F}} A$.

If \mathcal{F} is good (and satisfies a simple technical condition) then the **SDP** can also be recast in an equivalent way.

Proposition 4.2. Let $\mathcal{F} \in \text{GR}$ be such that $k \cap B \leq_{\mathcal{F}} B$ for every $k \in \omega$ and every $B \subseteq \mathbb{R}$ (we can require for instance that $\text{Lip}(2) \subseteq \mathcal{F}$). Then for every $A \subseteq \mathbb{R}$ the following are equivalent:

- i) A has the Strong Decomposition Property with respect to \mathcal{F} ;
- ii) if B is L -minimal in $[A]_{\mathcal{F}}$ then $B \leq_L \neg B$.

Moreover, B is either limit or successor of a nonselfdual pair with respect to \leq_L .

Proof. If $A = \emptyset$ or $A = \mathbb{R}$ neither i) nor ii) can hold, thus we can assume $A \neq \emptyset, \mathbb{R}$. If A has the Strong Decomposition Property with respect to \mathcal{F} , let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$ -bounded partition of \mathbb{R} such that $A \cap D_n <_{\mathcal{F}} A$ for every n and put $B_n = A \cap D_n$. Clearly we can not have that there is an $m \in \omega$ such that $B_n \leq_L B_m$ for every $n \in \omega$, otherwise $A \leq_{\mathcal{F}} B_m$ by condition (\star) , a contradiction! Therefore $\forall m \exists n (B_n \not\leq_L B_m)$ and hence $B = \bigoplus_n B_n$ is L -selfdual. Moreover $B \leq_L C$ for every $C \in [A]_{\mathcal{F}}$ since $B_n <_L C$ for every $n \in \omega$: on the other hand, $A \leq_{\mathcal{F}} B$ by condition (\star) again, hence B is L -minimal in $[A]_{\mathcal{F}}$.

Assume now that ii) holds. Recall that if C is an arbitrary subset of \mathbb{R} and $k \in \omega$, then $C_{[k]}$ denotes the set $\{x \in \mathbb{R} \mid k \cap x \in C\}$. It is a classical fact

that since B is \mathbb{L} -selfdual we have $B_{\lfloor k \rfloor} <_{\mathbb{L}} B$ for every $k \in \omega$, and hence by \mathbb{L} -minimality of B in $[A]_{\mathcal{F}}$ we have also $B_{\lfloor k \rfloor} <_{\mathcal{F}} B$. Our technical condition implies that $B \cap \mathbf{N}_{\langle k \rangle} \leq_{\mathcal{F}} B_{\lfloor k \rfloor}$, and since $\langle \mathbf{N}_{\langle k \rangle} \mid k \in \omega \rangle$ is always bounded in $\Delta_{\mathcal{F}}$ then B has the Strong Decomposition Property with respect to \mathcal{F} . But by Remark 4.1 this implies that A has the Strong Decomposition Property with respect to \mathcal{F} as well. The last part of the statement easily follows from our technical condition and the \mathbb{L} -minimality of B in $[A]_{\mathcal{F}}$, as if B is the successor with respect to $\leq_{\mathbb{L}}$ of an \mathbb{L} -selfdual set B' then $B \equiv_{\mathbb{L}} 0^{\mathbb{L}} B'$ (see e.g. [12] or [3] for a proof of this easy fact). \square

We will prove in Theorem 4.6 that the $\sigma\text{-BC}$ already implies the **SDP** (also in absence of the **PC** and of the other technical condition), but first we need the next proposition, which is a deep application of the Martin-Monk method and a further strengthening of Theorem 16 in [4] and of Theorem 5.3 in [8]. As for the other results of this kind, we will use the following lemma (probably due to Kuratowski, see Lemma 5.1 in [8] and the references given there).

Lemma 4.3 ($\text{ZF+AC}_{\omega}(\mathbb{R})$). *Let d be the usual metric on \mathbb{R} , τ the topology induced by d , and let ξ be any nonzero countable ordinal. For any family $\{D_n \mid n \in \omega\} \subseteq \Delta_{\xi}^0$ there is a metric d' on \mathbb{R} such that*

- i) (\mathbb{R}, τ') is Polish and zero-dimensional, where τ' is the topology induced by d' ;
- ii) τ' refines τ ;
- iii) each D_n is τ' -clopen;
- iv) there is a countable clopen basis \mathcal{B}' for τ' such that $\mathcal{B}' \subseteq \Delta_{\xi}^0$.

Proposition 4.4. *Assume that \mathcal{F} is of type I, II or III and has the $\sigma\text{-BC}$. Let $A \subseteq \mathbb{R}$ be such that $A \leq_{\mathcal{F}} \neg A$, $A \notin \Delta_{\mathcal{F}}$ and A is \mathbb{L} -minimal in its \mathcal{F} -degree. Then A has the Strong Decomposition Property with respect to \mathcal{F} .*

Proof. We start by considering the case in which \mathcal{F} is of type III, the other cases will be treated in a similar way. First observe that since $A \leq_{\mathcal{F}} \neg A$ we have $A \not\leq_{\mathcal{F}} A \cap D \iff A \cap D <_{\mathcal{F}} A$. Let $\xi < \omega_1$ be such that $\Delta_{\mathcal{F}} = \Delta_{<\xi}^0$, and let $f \in \mathcal{F}$ be any reduction of A into $\neg A$. Toward a contradiction, assume that for every $\Delta_{\mathcal{F}}$ -bounded partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} there is some $n_0 \in \omega$ such that $A \leq_{\mathcal{F}} A \cap D_{n_0}$. We will construct three sequences⁴

$$\langle C_n \mid n \in \omega \rangle, \langle d_n \mid n \in \omega \rangle, \langle f_n \mid n \in \omega \rangle$$

such that for every $n \in \omega$:

- i) $C_n \in \Delta_{\mathcal{F}}$ and $A \leq_{\mathcal{F}} A \cap C_n$;
- ii) $f_n: \mathbb{R} \rightarrow C_n$ is such that $f_n^{-1}(A \cap C_n) = A$ (i.e. f_n reduces A to $A \cap C_n$), and hence also $f_n^{-1}(\neg A \cap C_n) = \neg A$;
- iii) d_n is a metric on \mathbb{R} such that the induced topology τ_n is zero-dimensional and Polish, refines all the previous τ_m 's ($m \leq n$), C_n is τ_n -clopen, and both $f_n: (\mathbb{R}, \tau_{n+1}) \rightarrow (\mathbb{R}, \tau_n)$ and $f_n \circ f: (\mathbb{R}, \tau_{n+1}) \rightarrow (\mathbb{R}, \tau_n)$ are continuous;
- iv) for every $m \leq n$ and every $x, y \in C_{n+1}$

$$(*) \quad d_m(g_m \circ \dots \circ g_n(x), g_m \circ \dots \circ g_n(y)) < 2^{-n}$$

where for each i either $g_i = f_i \circ f \upharpoonright C_{i+1}$ or $g_i = f_i \upharpoonright C_{i+1}$.

Observe that by ii) we have that $f_n \circ f: \mathbb{R} \rightarrow C_n$ is such that

$$\forall x \in C_{n+1} (x \in A \cap C_{n+1} \iff f_n \circ f(x) \in \neg A \cap C_n),$$

⁴The major difference from the present argument and the proof of Theorem 5.3 in [8] is that in this case we will not require that $C_{n+1} \subseteq C_n$, so that we will have to use some special f_n 's (which “jump” from one C_n into the next one) rather than the identity function.

and that $f_n: \mathbb{R} \rightarrow C_n$ is such that

$$\forall x \in C_{n+1} (x \in A \cap C_{n+1} \iff f_n(x) \in A \cap C_n).$$

Having these sequences, we will be able to construct a flip-set (Wadge-reducible to A) using essentially the same argument contained in the proof of Theorem 16 in [4]. For every $z \in {}^\omega 2$ put $g_n^z = f_n \circ f$ if $z(n) = 1$, and $g_n^z = f_n$ otherwise. For every $n \in \omega$ choose some $y_{n+1} \in C_{n+1}$, and for every $m \leq n$ put $x_{n,m}^z = g_m^z \circ \dots \circ g_n^z(y_{n+1}) \in C_m$. If we fix m we get that $g_m^z(x_{n,m+1}^z) = x_{n,m}^z$ for every $n > m$, and that $\{x_{n,m}^z \mid n \geq m\} \subseteq C_m$ is a Cauchy sequence with respect to d_m by (*). Therefore we can put $x_m^z = \lim_{n \rightarrow \infty} x_{n,m}^z$ and notice that $x_m^z \in C_m$ by the fact that C_m is τ_m -closed, and that $g_m^z(x_{m+1}^z) = x_m^z$ by continuity of g_m^z . Now it is easy to verify that $F = \{z \in {}^\omega 2 \mid x_0^z \in A\}$ is a flip-set.

The construction of the required sequences will be carried out by induction on n . To reach this goal we will construct also two auxiliary sequences

$$\langle \mathcal{P}_n \mid n \in \omega \rangle, \langle \mu_n \mid n \in \omega \rangle$$

such that:

- 1) μ_n is an increasing sequence of ordinals smaller than ξ and f_n is a $\Delta_{\mu_n}^0$ -function;
- 2) τ_n admits a countable basis $\mathcal{B}_n \subseteq \Delta_{\mu_n}^0$;
- 3) $\mathcal{P}_n = \langle D_m^n \mid m \in \omega \rangle$ is a $\Delta_{\mu_n}^0$ -partition of \mathbb{R} (in particular is bounded in $\Delta_{\mathcal{F}}$), \mathcal{P}_{n+1} refines \mathcal{P}_n , $C_n = D_m^n$ for some m , and each D_m^n is τ_n -clopen.

At stage n we will define C_n , \mathcal{P}_n , f_n together with d_{n+1} and μ_{n+1} . First let $C_0 = \mathbb{R}$, \mathcal{P}_0 be defined by $D_0^0 = \mathbb{R}$ and $D_{m+1}^0 = \emptyset$, $f_0 = \text{id}$, $\mu_0 = 1$, and d_0 be the usual metric on \mathbb{R} . By $\sigma\text{-BC}$ there is some $\mu_1 < \xi$ such that $\{f^{-1}(\mathbf{N}_s) \mid s \in {}^{<\omega}\omega\} \subseteq \Delta_{\mu_1}^0$, and we can let d_1 be the metric obtained applying Lemma 4.3 to this collection of sets (so that $f = f_0 \circ f: (\mathbb{R}, \tau_1) \rightarrow (\mathbb{R}, \tau_0)$ is continuous). For the inductive step we need the following claim, which is analogous to Claim 5.3.1 of [8].

Claim 4.4.1. Let $D \subseteq \mathbb{R}$ be in Δ_{μ}^0 (for some $\mu < \xi$). If $A \leq_{\mathcal{F}} A \cap D$ then there is $g \in D_{\mu}$ such that $g: \mathbb{R} \rightarrow D$ and g reduces A to $A \cap D$.

Proof of the Claim. We can assume $D \neq \emptyset, \mathbb{R}$ and $\neg A \cap D \neq \emptyset$, as if $D = \mathbb{R}$ then we can simply take g to be the identity, while if $D = \emptyset$ or $D \subseteq A$ then $A \leq_{\mathcal{F}} A \cap D = D$ would contradict $A \notin \Delta_{\mathcal{F}}$. By the observation above we have that $A \not\leq_{\mathcal{F}} A \cap D \iff A \cap D <_{\mathcal{F}} A$, and by Lemma 2.1 and L-minimality of A in its \mathcal{F} -degree we have that $A \cap D <_{\mathcal{F}} A \iff A \cap D <_{\text{L}} A$. Thus $A \leq_{\mathcal{F}} A \cap D$ implies that either $A \leq_{\text{L}} A \cap D$ or, by SLO^{L} , $\neg A \leq_{\text{L}} A \cap D$. If the second alternative holds, then since $A \cap D \leq_{D_{\mu}} A$ (see Lemma 4.4 in [8]) one also has $A \leq_{D_{\mu}} \neg A$: thus in every case $A \leq_{D_{\mu}} A \cap D$. Let $g' \in D_{\mu}$ be a witness of this fact. Let $k \in D_{\mu}$ be defined by $k(x) = x$ if $x \in D$ and $k(x) = y$ otherwise, where y is any fixed point in $\neg A \cap D$. Letting $g = k \circ g'$ it is easy to check that our claim holds. \square *Claim*

Now suppose to have constructed all the sequences until stage n , that is C_i , \mathcal{P}_i , f_i , d_j and μ_j for $i \leq n$ and $j \leq n+1$. Recall also from Claim 5.3.2 of [8] that for every $m \leq n$ there is a $\Delta_{\mu_m}^0$ -partition $\{C_m^i \mid i \in \omega\}$ of \mathbb{R} such that d_m -diam(C_m^i) $< 2^{-n}$ and C_m^i is τ_m -clopen for every $i \in \omega$. Fix $s \in {}^{n+1}2$ and let g_k^s be defined (for every $k \leq n$) by

$$g_k^s = \begin{cases} f_k \circ f & \text{if } s(n) = 1 \\ f_k & \text{if } s(n) = 0. \end{cases}$$

Let $\langle D_{i,s}^0 \mid i \in \omega \rangle$ be an enumeration of

$$\{(g_0^s \circ \dots \circ g_n^s)^{-1}(C_0^i) \cap D_m^n \mid i, m \in \omega\}$$

and for $k < n$ let $\langle D_{i,s}^{k+1} \mid i \in \omega \rangle$ be an enumeration of

$$\{(g_{k+1}^s \circ \dots \circ g_n^s)^{-1}(C_{k+1}^i) \cap D_{j,s}^k \mid i, j \in \omega\}.$$

Arguing by induction on $k \leq n$, it is not hard to see that $\langle D_{i,s}^n \mid i \in \omega \rangle$ is a $\Delta_{\mu_{n+1}}^0$ -partition which refines \mathcal{P}_n , and that each $D_{i,s}^n$ is τ_{n+1} -clopen since, by induction on $k < n$ again, one can prove that $g_k^s \circ \dots \circ g_n^s: (\mathbb{R}, d_{n+1}) \rightarrow (\mathbb{R}, d_k)$ is continuous (and the τ_{n+1} -clopen sets are contained in $\Delta_{\mu_{n+1}}^0$, as $\mathcal{B}_{n+1} \subseteq \Delta_{\mu_{n+1}}^0$ by inductive hypothesis).

Now fix an enumeration $\langle s_l \mid l < 2^{n+1} \rangle$ of 2^{n+1} and inductively repeat the argument above but using $\langle D_{i,s_l}^n \mid i \in \omega \rangle$ instead of \mathcal{P}_n at stage $l+1$. Let $\mathcal{P}_{n+1} = \langle D_m^{n+1} \mid m \in \omega \rangle$ be the final partition of \mathbb{R} obtained at stage 2^{n+1} , and observe that one has again that $D_m^{n+1} \in \Delta_{\mu_{n+1}}^0$ and that D_m^{n+1} is τ_{n+1} -clopen for every $m \in \omega$. Choose $\bar{m} \in \omega$ such that $A \leq_{\mathcal{F}} A \cap D_{\bar{m}}^{n+1}$ (such an \bar{m} must exist by our assumption, since \mathcal{P}_{n+1} is a $\Delta_{\mathcal{F}}$ -bounded partition of \mathbb{R}), put $C_{n+1} = D_{\bar{m}}^{n+1}$, and let f_{n+1} be the function obtained applying Claim 4.4.1 to C_{n+1} . We claim that there is some $\mu_{n+2} \geq \mu_{n+1}$ smaller than ξ such that both

$$\{f_{n+1}^{-1}(B) \mid B \in \mathcal{B}_{n+1}\} \subseteq \Delta_{\mu_{n+2}}^0 \quad \text{and} \quad \{(f_{n+1} \circ f)^{-1}(B) \mid B \in \mathcal{B}_{n+1}\} \subseteq \Delta_{\mu_{n+2}}^0.$$

In fact, the first part is obvious (since f_{n+1} is a $\Delta_{\mu_{n+1}}^0$ -function and $\mathcal{B}_{n+1} \subseteq \Delta_{\mu_{n+1}}^0$). For the second part, since $\{f_{n+1}^{-1}(B) \mid B \in \mathcal{B}_{n+1}\} \subseteq \Delta_{\mu_{n+1}}^0$ is countable and bounded in $\Delta_{\mathcal{F}}$, by the σ -BC there must be some $\nu < \xi$ such that

$$\{f^{-1}(f_{n+1}^{-1}(B)) \mid B \in \mathcal{B}_{n+1}\} \subseteq \Delta_{\nu}^0.$$

Put $\mu_{n+2} = \max\{\mu_{n+1}, \nu\}$: it is easy to check that μ_{n+2} is as required.

Finally, apply Lemma 4.3 to the collection

$$\{f_{n+1}^{-1}(B), (f_{n+1} \circ f)^{-1}(B) \mid B \in \mathcal{B}_{n+1}\} \subseteq \Delta_{\mu_{n+2}}^0$$

to get d_{n+2} with the desired properties (in particular, we have that both $f_{n+1} \circ f: (\mathbb{R}, d_{n+1}) \rightarrow (\mathbb{R}, d_n)$ and $f_{n+1}: (\mathbb{R}, d_{n+1}) \rightarrow (\mathbb{R}, d_n)$ are continuous). It is not hard to check that the sequences inductively constructed in this way satisfy all the conditions required, and this concludes the proof for the case when \mathcal{F} is of type III.

Now let us consider the other possibilities for the set of reductions \mathcal{F} : if \mathcal{F} is of type I we can use the same argument as above but avoiding to construct the μ_n 's, letting d_n be the usual metric on \mathbb{R} for every $n \in \omega$ (thus dropping essentially condition iii), and constructing the partitions \mathcal{P}_n in such a way that for each $n \in \omega$ there is some k_n such that each element of \mathcal{P}_n , and in particular C_n , is L -reducible to $\mathbf{N}_{0(k_n)}$ (the collection $\Delta_{k_n} = \{A \subseteq \mathbb{R} \mid A \leq_{\mathsf{L}} \mathbf{N}_{0(k_n)}\}$ can be easily seen to be closed under finite intersections and unions)⁵. Finally, if \mathcal{F} is of type II one can repeat the argument above (in a slightly simpler way) taking advantage of the fact that *every* countable family of Δ_{ξ}^0 sets is bounded in Δ_{ξ}^0 . \square

Remark 4.5. We can completely remove the hypothesis that A is L -minimal in its \mathcal{F} -degree and reprove Proposition 4.4 assuming only $\mathsf{ZF} + \mathsf{AC}_{\omega}(\mathbb{R}) + \neg \mathsf{FS}$ (thus giving essentially a direct proof of Theorem 4.6 under a weaker axiomatization) whenever \mathcal{F} satisfies the following property (which is a consequence of **PC**): if $D \in \Delta_{\mathcal{F}}$ and f is a constant function then $\text{id} \upharpoonright D \cup f \upharpoonright \neg D$ is in \mathcal{F} . This is because in this case we can compose any reduction $f \in \mathcal{F}$ of A into $A \cap D$ with the function k defined in the proof of Claim 4.4.1 to get that if $A \leq_{\mathcal{F}} A \cap D$ then there is some $g \in \mathcal{F}$ such that $g: \mathbb{R} \rightarrow D$ and g reduces A to $A \cap D$. This fact can then be used to construct the f_n 's and conclude the argument exactly in the same way. About the construction,

⁵One has also to modify the statement of Claim 4.4.1 in the following way: “Let $D \subseteq \mathbb{R}$ be in Δ_{k_n} (for some $k_n \in \omega$). If $A \leq_{\mathcal{F}} A \cap D$ then there is a $g: \mathbb{R} \rightarrow D$ such that g is a Δ_{k_n} -function which reduces A to $A \cap D$ ”.

one should just be careful in the inductive step, and check that an ordinal μ_{n+2} with the desired properties exists because both f_{n+1} and f are σ -bounded in $\Delta_{\mathcal{F}}$, and the composition of σ -bounded functions is still σ -bounded.

Now we are ready to prove the Strong Decomposition Theorem.

Theorem 4.6. *Let \mathcal{F} be a Borel set of reductions which satisfies the σ -BC. Then \mathcal{F} has the **SDP**.*

Proof. Assume first that \mathcal{F} is of type I, II or III. Let $A \leq_{\mathcal{F}} \neg A \notin \Delta_{\mathcal{F}}$ and let B be L -minimal in $[A]_{\mathcal{F}}$: then B has the Strong Decomposition Property with respect to \mathcal{F} by Proposition 4.4, which by Remark 4.1 implies that A has the Strong Decomposition Property with respect to \mathcal{F} as well.

Now assume that \mathcal{F} is not of type I–III, i.e. that $\Delta_{<\xi}^0 \subsetneq \Delta_{\mathcal{F}} \subsetneq \Delta_{\xi}^0$ for some countable ξ (notice that in this case we will not use the σ -BC). By Proposition 3.3 of [8] we have that $\mathcal{F} \subseteq \mathsf{D}_{\xi}$, thus if $A \leq_{\mathcal{F}} \neg A$ we have also $A \leq_{\mathsf{D}_{\xi}} \neg A$. By the **SDP** for D_{ξ} , there must be a Δ_{ξ}^0 -partition $\langle D'_n \mid n \in \omega \rangle$ of \mathbb{R} such that $A \cap D'_n <_{\mathsf{D}_{\xi}} A$ for every n . This partition can be refined to a $\bigcup_{\mu < \xi} \Pi_{\mu}^0$ -partition $\langle D_n \mid n \in \omega \rangle$ of \mathbb{R} with the same property, that is such that $A \cap D_n <_{\mathsf{D}_{\xi}} A$ for every n . But $\bigcup_{\mu < \xi} \Pi_{\mu}^0$ is easily seen to be bounded in $\Delta_{\mathcal{F}}$, and $A \cap D_n <_{\mathcal{F}} A$ by **SLO**. \square

The Strong Decomposition Theorem (together with part *c*) of Lemma 2.3) implies that $A \leq_{\mathcal{F}} \neg A$ if and only if A has the Strong Decomposition Property with respect to \mathcal{F} , thus if \mathcal{F} is good we can adjoin the condition $A \leq_{\mathcal{F}} \neg A$ to the equivalents of Proposition 4.2. Moreover, as a corollary of Theorem 4.6 one gets also that if \mathcal{F} is a good Borel reducibility then at limit levels of uncountable cofinality there is a nonselfdual pair.

Corollary 4.7. *Let \mathcal{F} be a good Borel set of reductions and let $[A]_{\mathcal{F}}$ be a selfdual limit degree. Then $[A]_{\mathcal{F}}$ is of countable cofinality.*

Proof. Let $\langle D_n \mid n \in \omega \rangle$ be a $\Delta_{\mathcal{F}}$ -bounded partition of \mathbb{R} such that $A \cap D_n <_{\mathcal{F}} A$ and

$$(\dagger) \quad \forall n \in \omega \exists m \in \omega (A \cap D_n <_{\mathcal{F}} A \cap D_m)$$

(such a partition must exist by Theorem 4.6 and by the fact that $[A]_{\mathcal{F}}$ is limit): then $\mathcal{A} = \{[A \cap D_n]_{\mathcal{F}} \mid n \in \omega\}$ witnesses that $[A]_{\mathcal{F}}$ is of countable cofinality (use condition (\star) and the fact that if $A \cap D_n \leq_{\mathcal{F}} B$ for every n then $A \cap D_n <_{\mathsf{L}} B$ by (\dagger) and Lemma 2.1). \square

The Strong Decomposition Theorem implies also that we can compare good Borel reducibilities with respect to the degree-structures induced by them.

Theorem 4.8. *Let \mathcal{F} and \mathcal{G} be two Borel sets of reductions such that \mathcal{G} has the **SDP**, \mathcal{F} satisfies the **PC**, and $\Delta_{\mathcal{G}} \subseteq \Delta_{\mathcal{F}}$. Then for every $A, B \subseteq \mathbb{R}$*

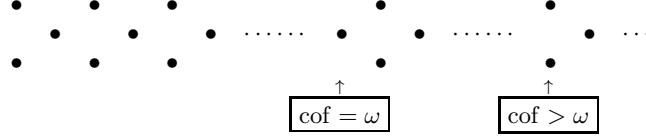
$$A \leq_{\mathcal{G}} B \Rightarrow A \leq_{\mathcal{F}} B.$$

In particular, if \mathcal{F} and \mathcal{G} are good Borel reducibilities then $\mathcal{F} \simeq \mathcal{G}$ if and only if $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$.

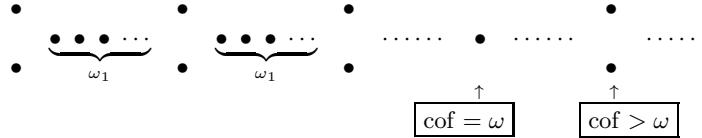
The proof is identical to the one of Theorem 4.7 in [8] — the only obvious modification is that we have to use **SDP** instead of **DP**. Using Theorem 4.8, one can now obtain the dichotomy theorem for good Borel reducibilities (which is simply a more detailed recasting of Theorem 1.1).

Theorem 4.9. *Let \mathcal{F} be a good Borel reducibility. Then one of the following holds:*

i) \mathcal{F} induces a Wadge-like degree-structure:



ii) \mathcal{F} induces a Lipschitz-like degree-structure:



In particular, the first alternative holds if \mathcal{F} is of type II, while the second alternative holds if \mathcal{F} is either of type I or of type III.

The proof of this theorem can be obtained by choosing some “canonical” representative for each equivalence class induced by the equivalence relation \simeq on GR , and by studying the degree-structure induced by it. These examples are, respectively: Lip for the collection of the good Borel reducibilities of type I, D_ξ or Bor for the \mathcal{F} ’s of type II such that $\Delta_{\mathcal{F}} = \Delta_\xi^0$ (for each $\xi \leq \omega_1$), and the chain of reductions $\bigcup_{\mu < \xi} D_\mu$ for the \mathcal{F} ’s of type III such that $\Delta_{\mathcal{F}} = \Delta_{<\xi}^0$ (for every countable limit ξ). The degree-structures of Bor and D_ξ have already been determined in [8], while the degree-structures of Lip and of the chains of reductions will be determined in the next section of this paper (one can check that all these results are coherent with the description given in Theorem 4.9). Therefore it will be enough to apply Theorem 4.8, with \mathcal{G} being the suitable “canonical” representative (i.e. the canonical example such that $\Delta_{\mathcal{F}} = \Delta_{\mathcal{G}}$), to get the result for an arbitrary good Borel reducibility \mathcal{F} .

5. GOOD BOREL REDUCIBILITIES OF TYPE I AND III

In this section we will analyze the degree-structures induced by Lip and by (regular) chains of reductions, showing in particular that they are all Lipschitz-like. This will complete the proof of Theorem 4.9.

5.1. Lipschitz functions. First we want to prove that Lip is a *good* Borel reducibility of type I, and this practically amounts to compute that

$$\Delta_{\text{Lip}} = \bigcup_{0 \neq n \in \omega} [\mathbf{N}_{0(n)}]_{\text{L}} \cup \{\emptyset, \mathbb{R}\} = \bigcup_{s \in {}^{<\omega}\omega} [\mathbf{N}_s]_{\text{L}} \cup \{\emptyset\}.$$

One direction is obvious, so we will just prove $\Delta_{\text{Lip}} \subseteq \bigcup_{0 \neq n \in \omega} [\mathbf{N}_{0(n)}]_{\text{L}} \cup \{\emptyset, \mathbb{R}\}$. Let $\emptyset \neq A \in \Delta_{\text{Lip}}$: by definition there are $f \in \text{Lip}$ and $n \in \omega$ such that $f \in \text{Lip}(2^n)$ and $f^{-1}(\mathbf{N}_{\langle 0 \rangle}) = A$. We want to show that $S = \{s \in {}^{n+1}\omega \mid f(\mathbf{N}_s) \subseteq \mathbf{N}_{\langle 0 \rangle}\}$ is such that $A = \bigcup_{s \in S} \mathbf{N}_s$: since the set on the right of the equation is clearly L -reducible to $\mathbf{N}_{0(n+1)}$, this will finish the proof. Clearly $\bigcup_{s \in S} \mathbf{N}_s \subseteq A$. For the other direction, pick any $x \in A$: being f a reduction of A into $\mathbf{N}_{\langle 0 \rangle}$, $f(x) \in \mathbf{N}_{\langle 0 \rangle}$. Since $f \in \text{Lip}(2^n)$, $d(f(x), f(y)) \leq 2^{-1}$ for every $y \in \mathbf{N}_{x \upharpoonright (n+1)}$, which means $f(\mathbf{N}_{x \upharpoonright (n+1)}) \subseteq \mathbf{N}_{\langle 0 \rangle}$: but then $x \upharpoonright (n+1) \in S$ and hence $x \in \bigcup_{s \in S} \mathbf{N}_s$.

Since we have just proved that $\text{Lip} \in \text{GR}$, to determine the degree-structure induced by Lip we have only to understand what happens after a selfdual degree. Given any set $A \subseteq \mathbb{R}$ define

$$s_{\text{Lip}}(A) = \bigoplus_n 0^{(n)} \cap A.$$

We want to prove that if $A \leq_{\text{Lip}} \neg A$ then $[s_{\text{Lip}}(A)]_{\text{Lip}}$ is selfdual and is the immediate successor of $[A]_{\text{Lip}}$. This will prove that after a selfdual Lip -degree there is

always another selfdual Lip -degree, and that Lip induce a degree-structure which is Lipschitz-like.

Proposition 5.1. *Let $A \subseteq \mathbb{R}$ be Lip -selfdual. Then $s_{\text{Lip}}(A) \leq_{\text{Lip}} \neg s_{\text{Lip}}(A)$, $A <_{\text{Lip}} s_{\text{Lip}}(A)$ and there is no B such that $A <_{\text{Lip}} B <_{\text{Lip}} s_{\text{Lip}}(A)$.*

Proof. Let A be L -minimal in its Lip -degree and observe that one has $A \leq_{\text{L}} \neg A$ by Proposition 4.2 (note that obviously $\text{Lip}(2) \subseteq \text{Lip}$). This implies that $A <_{\text{L}} 0^\frown A <_{\text{L}} \dots <_{\text{L}} 0^{(n)} \frown A <_{\text{L}} \dots$, and hence that $s_{\text{Lip}}(A) \leq_{\text{L}} \neg s_{\text{Lip}}(A)$. Moreover it is clear that for every $n \in \omega$ we have $A \leq_{\text{L}} 0^{(n)} \frown A$ and that $0^{(n)} \frown A \leq_{\text{Lip}} A$ via a function $f \in \text{Lip}(2^n)$. If $B <_{\text{Lip}} s_{\text{Lip}}(A)$ we have that $B <_{\text{L}} s_{\text{Lip}}(A)$ by Lemma 2.1, which in turn implies $B \leq_{\text{L}} 0^{(n)} \frown A$ for some $n \in \omega$: hence $B \leq_{\text{Lip}} A$. Therefore it remains only to prove that $s_{\text{Lip}}(A) \not\leq_{\text{Lip}} A$. Toward a contradiction, assume that there is $f \in \text{Lip}$ such that $f^{-1}(A) = s_{\text{Lip}}(A)$, and let n be the smallest natural number such that $f \in \text{Lip}(2^n)$, so that $f(\mathbf{N}_{0^{(n+1)}}) \subseteq \mathbf{N}_{\langle k \rangle}$ for some $k \in \omega$. Let g be defined by $g(x) = f(x)$ if $x \in \mathbf{N}_{0^{(n+1)}}$ and $g(x) = (k+1)^\frown \vec{0}$ otherwise: then $g \in \text{Lip}(2^{n+1})$ and reduces $0^{(n+1)} \frown A$ to $A \cap \mathbf{N}_{\langle k \rangle}$. But it is easy to check that $A \cap \mathbf{N}_{\langle k \rangle} \leq_{\text{Lip}} A_{\lfloor k \rfloor}$ and $A_{\lfloor k \rfloor} <_{\text{L}} A$: therefore, by L -minimality of A in its Lip -degree we would have that

$$0^{(n+1)} \frown A \leq_{\text{Lip}} A_{\lfloor k \rfloor} <_{\text{Lip}} A,$$

a contradiction! \square

The definition of the successor operator s_{Lip} , allows also to obtain a way to construct the Lip -degrees from the L -degrees. In fact, if $[A]_{\text{Lip}}$ is nonselfdual, then $[A]_{\text{Lip}} = [A]_{\text{L}}$ by Theorem 2.2, while if $A \leq_{\text{L}} \neg A$ and A is L -minimal in its Lip -degree, then $[A]_{\text{Lip}}$ is exactly $\bigcup_{n \in \omega} [0^{(n)} \frown A]_{\text{L}}$.

As an application of Theorem 4.8, let us now consider the set of the uniformly continuous functions (which will be denoted by UCont): it turns out (perhaps rather surprisingly, since uniform continuity is just a weak ‘‘refinement’’ of continuity) that UCont is equivalent to Lip (rather than to W), and thus gives a hierarchy of degrees which is Lipschitz-like. In fact, one can easily check that UCont is a good Borel reducibility and that $\Delta_{\text{UCont}} = \bigcup_{s \in <_{\omega\omega}} [\mathbf{N}_s]_{\text{L}} \cup \{\emptyset\}$: \mathbf{N}_s is reducible to $\mathbf{N}_{\langle 0 \rangle}$ via a function in $\text{Lip}(2^{\text{lh}(s)}) \subseteq \text{UCont}$, while if f is uniformly continuous then there must be some $m \in \omega$ such that for every $x, y \in \mathbb{R}$

$$d(x, y) \leq 2^{-m} \Rightarrow d(f(x), f(y)) \leq 2^{-1},$$

and thus, in particular, f can not reduce $\bigoplus_n \mathbf{N}_{0^{(n)}}$ to $\mathbf{N}_{\langle 0 \rangle}$ (the argument is similar to the one used in Proposition 5.1). This proves also that $\Delta_{\text{Lip}} = \Delta_{\text{UCont}}$, and that UCont is of type I: therefore $\text{Lip} \simeq \text{UCont}$ by Theorem 4.8. Moreover it is not hard to check that UCont is maximal among the good Borel reducibilities of type I, since the fact that \mathcal{F} is of type I and satisfies the $\sigma\text{-BC}$ implies $\mathcal{F} \subseteq \text{UCont} = \text{UCont}$ is exactly the collection of all σ -bounded Δ_{Lip} -functions.

5.2. Chains of reductions. A (countable) *chain of (Borel-amenable sets of) reductions* is simply any sequence $\vec{\mathcal{F}} = \langle \mathcal{F}_n \mid n \in \omega \rangle$ of Borel-amenable sets of reductions. To each chain of reductions we will associate the unique sequence of ordinals $\langle \mu_n \mid n \in \omega \rangle$ such that $1 \leq \mu_n \leq \omega_1$ and $\Delta_{\mathcal{F}_n} = \Delta_{\mu_n}^0$ for every $n \in \omega$, which will be called the *type of $\vec{\mathcal{F}}$* . Moreover we will say that $\vec{\mathcal{F}}$ is of *rank* ω_1 if $\mu_n = \omega_1$ for some $n \in \omega$, and of *rank* $1 \leq \xi < \omega_1$ if $\mu_n < \omega_1$ for every $n \in \omega$ and $\xi = \sup\{\mu_n + 1 \mid n \in \omega\}$. A chain of reductions will be called *regular* if each \mathcal{F}_n is saturated and $\mathcal{F}_n \subsetneq \mathcal{F}_{n+1}$ for every n (in particular, the rank ξ of $\vec{\mathcal{F}}$ must be

countable and limit⁶). Note that in this case $\bigcup_n \mathcal{F}_n = \bigcup_{\mu < \xi} D_\mu$ is a Borel set of reductions, and since

$$\Delta_{\bigcup_n \mathcal{F}_n} = \bigcup_n \Delta_{\mathcal{F}_n} = \bigcup_n \Delta_{\mu_n}^0 = \Delta_{<\xi}^0$$

one can check that $\bigcup_n \mathcal{F}_n$ is good and of type III: thus, as we have already pointed out, regular chains of reductions provide a canonical way to construct good Borel reducibilities of type III (one for each possible characteristic set). From now onward, we will fix some limit $\xi < \omega_1$ and consider a regular chain of reductions $\vec{\mathcal{F}} = \langle \mathcal{F}_n \mid n \in \omega \rangle$ of rank ξ . By Corollary 4.7, in order to describe the structure of degrees induced by⁷ $\leq_{\vec{\mathcal{F}}}$ we have only to determine what happens after a selfdual degree: this can be done using the following proposition about Borel-amenable sets of reductions.

Proposition 5.2. *Let \mathcal{G} and \mathcal{G}' be two Borel-amenable sets of reductions such that $\Delta_{\mathcal{G}} \subsetneq \Delta_{\mathcal{G}'}$ (i.e. such that \mathcal{G} is of level strictly smaller than \mathcal{G}'). Let $A \leq_{\mathcal{G}} \neg A$ and B be a (nonselfdual) successor of A with respect to $\leq_{\mathcal{G}}$: then $B \leq_{\mathcal{G}'} A$. In particular, if μ is the level of \mathcal{G} and $A \leq_{\mathcal{G}} \neg A$, then $\Sigma^\mu(A) \leq_{\mathcal{G}'} A$ and $\Pi^\mu(A) \leq_{\mathcal{G}'} A$.*

Proof. If \mathcal{G} and B are as above then either $B \equiv_{\mathcal{G}} \Sigma^\mu(A)$ or $B \equiv_{\mathcal{G}} \Pi^\mu(A) \equiv_{\mathcal{G}} \neg \Sigma^\mu(A)$, and since $\Delta_{\mathcal{G}} \subseteq \Delta_{\mathcal{G}'}$ implies $A \leq_{\mathcal{G}} B \Rightarrow A \leq_{\mathcal{G}'} B$ for every $A, B \subseteq \mathbb{R}$ (by Theorem 4.8), it is enough to prove $\Sigma^\mu(A) \leq_{\mathcal{G}'} A$. Let P_n and R_μ be the sets used to define the operation Σ^μ , and define $F_0 = \{x \in \mathbb{R} \mid \pi_0(x) \in P_0\}$ and $F_{n+1} = \{x \in \mathbb{R} \mid \pi_{2(n+1)}(x) \in P_{n+1} \wedge \forall i \leq n (\pi_{2i}(x) \notin P_i)\}$. Clearly, every $F_n \in \Delta_\mu^0 \subseteq \Delta_{\mathcal{G}'}$, and since $R_\mu \in \Pi_\mu^0$ and $\Delta_\mu^0 \subsetneq \Delta_{\mathcal{G}'}$, we have also $R_\mu \in \Delta_{\mathcal{G}'}$ (as $\Sigma_\mu^0 \cup \Pi_\mu^0 \subseteq \Delta_{\mathcal{G}'}$ by Borel-determinacy). On each of these sets we can continuously reduce $\Sigma^\mu(A)$ to A using π_{2n+1} on the F_n 's and a constant function with value $\bar{y} \notin A$ on R_μ ($A \neq \mathbb{R}$ as $A \leq_{\mathcal{G}} \neg A$), hence $\Sigma^\mu(A) \leq_{D_{\mu'}^W} A$, where μ' is the level of \mathcal{G}' . But since $\mathcal{G}' \simeq D_{\mu'}^W$ we are done. \square

Theorem 5.3. *If $A \leq_{\vec{\mathcal{F}}} \neg A$ then there is some $B \leq_{\vec{\mathcal{F}}} \neg B$ with the property that $A <_{\vec{\mathcal{F}}} B$ and there is no C such that $A <_{\vec{\mathcal{F}}} C <_{\vec{\mathcal{F}}} B$. Thus after an $\vec{\mathcal{F}}$ -selfdual degree there is another $\vec{\mathcal{F}}$ -selfdual degree.*

Proof. Taking A to be \mathbb{L} -minimal in $[A]_{\vec{\mathcal{F}}}$, by Proposition 4.2 and the fact that $\vec{\mathcal{F}}$ has the **SDP** we can assume $A \leq_{\mathbb{L}} \neg A$ (hence, in particular, $A \leq_{\mathcal{F}_n} \neg A$ for every $n \in \omega$). Let $\langle \mu_n \mid n \in \omega \rangle$ be the type of $\vec{\mathcal{F}}$ and define the *successor operator* $s_{\vec{\mathcal{F}}}$ by letting

$$B = s_{\vec{\mathcal{F}}}(A) = \bigoplus_n \Sigma^{\mu_n}(A).$$

Clearly $A \leq_{\mathbb{L}} s_{\vec{\mathcal{F}}}(A)$, and if $C <_{\vec{\mathcal{F}}} s_{\vec{\mathcal{F}}}(A)$ then we have also $C <_{\mathbb{L}} s_{\vec{\mathcal{F}}}(A)$, which implies $C \leq_{\mathbb{L}} \Sigma^{\mu_n}(A)$ for some $n \in \omega$: but since $\Sigma^{\mu_n}(A) \leq_{\mathcal{F}_{n+1}} A$ by Proposition 5.2, we have also $C \leq_{\vec{\mathcal{F}}} A$. Finally, the fact that $s_{\vec{\mathcal{F}}}(A) \leq_{\vec{\mathcal{F}}} \neg s_{\vec{\mathcal{F}}}(A)$ will follow from the fact that $\Sigma^{\mu_n}(A) <_{\mathbb{L}} \Sigma^{\mu_{n+1}}(A)$ for every $n \in \omega$ (since this implies $s_{\vec{\mathcal{F}}}(A) \leq_{\mathbb{L}} \neg s_{\vec{\mathcal{F}}}(A)$). To see this, recall that $\Sigma^{\mu_n}(A) \leq_{\mathcal{F}_{n+1}} A$ while $A <_{\mathcal{F}_{n+1}} \Sigma^{\mu_{n+1}}(A)$, which implies $\Sigma^{\mu_n}(A) <_{\mathcal{F}_{n+1}} \Sigma^{\mu_{n+1}}(A)$: hence $\Sigma^{\mu_n}(A) <_{\mathbb{L}} \Sigma^{\mu_{n+1}}(A)$ by Lemma 2.1. \square

In particular, Theorem 2.2, Corollary 4.7 and Theorem 5.3 implies that the degree-structure induced by any regular chain of reductions $\vec{\mathcal{F}}$ (i.e. by the preorder $\leq_{\vec{\mathcal{F}}}$) is Lipschitz-like.

⁶It is easy to check that each chain of reductions is equivalent either to a Borel-amenable set of reductions (if it has successor or uncountable rank) or to a regular chain of reductions with the same rank.

⁷For simplicity of notation, from now on we will systematically identify $\vec{\mathcal{F}}$ with $\bigcup_n \mathcal{F}_n$ when there is no possibility of misunderstanding.

6. BAIRE REDUCTIONS

Let \mathcal{B}_α (for $\alpha < \omega_1$) denote the set of all Baire class α functions from \mathbb{R} into itself, i.e. the set of all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f^{-1}(U) \in \Sigma_{\alpha+1}^0$ for every open set U . Clearly $D_1 = \mathcal{B}_0 \subseteq \mathcal{B}_\alpha \subseteq \text{Bor}$ for every $\alpha < \omega_1$, and $\mathcal{B}_\mu \subseteq \mathcal{B}_\nu$ if and only if $\mu \leq \nu$. Moreover it is well known that the Baire class α functions provides a stratification of Bor in ω_1 -many levels which is alternative to the one induced by Δ_ξ^0 -functions, thus it is quite natural to try to study the reducibilities induced by the Baire class functions (of some level). Unfortunately, if $\alpha \neq 0$ then \mathcal{B}_α is not a set of reductions as it is not closed under composition: in fact, it is easy to check that if $f \in \mathcal{B}_\mu$ and $g \in \mathcal{B}_\nu$ then $g \circ f \in \mathcal{B}_{\mu+\nu}$ and, moreover, there are such an f and g for which $g \circ f \notin \mathcal{B}_\eta$ for any $\eta < \mu+\nu$. Nevertheless, we can exactly compute the closure under composition of \mathcal{B}_α by reversing the previous composition law, i.e. by showing that if $h \in \mathcal{B}_{\mu+\nu}$ then there are $f \in \mathcal{B}_\mu$ and $g \in \mathcal{B}_\nu$ such that $h = g \circ f$. To obtain this computation we will use Lemma 4.3 together with the following crucial fact (for simplicity of notation we will put $\Delta_0^0 = \Delta_1^0$).

Lemma 6.1 (ZF + AC $_\omega(\mathbb{R})$). *For every nonzero $\mu, \nu < \omega_1$ with $\nu > 1$ and every $\mathcal{C} = \{C_n \mid n \in \omega\} \subseteq \Delta_{\mu+\nu}^0$ there is $\mathcal{B} = \{B_m \mid m \in \omega\} \subseteq \Delta_{\mu+1}^0$ such that $\mathcal{C} \subseteq \Delta_\nu^0(\tau')$ for every topology τ' for which $\mathcal{B} \subseteq \Delta_1^0(\tau')$.*

Proof. Clearly we can assume that \mathcal{C} is closed under complementation (if not simply adjoin $\neg C_n$ to \mathcal{C} for every n). We will prove the lemma by induction on ν , and the base of the induction and the successor case will be proved together. Assume $\nu = \eta + 1$ (with $\eta \geq 1$): by definition there must be a collection $\mathcal{D}' = \{D'_{n,k} \mid n, k \in \omega\} \subseteq \Pi_{\mu+\eta}^0$ such that $C_n = \bigcup_{k \in \omega} D'_{n,k}$ for every n , and by definition again there must be some $\mathcal{D} = \{D_{n,k,i} \mid n, k, i \in \omega\} \subseteq \Delta_{\mu+\eta}^0$ such that $D'_{n,k} = \bigcap_{i \in \omega} D_{n,k,i}$ for every n, k . Put $\mathcal{B} = \mathcal{D}$ if $\eta = 1$ or, in the other case, use the inductive hypothesis applied to \mathcal{D} to find some countable $\mathcal{B} \subseteq \Delta_{\mu+1}^0$ such that for every topology τ' on \mathbb{R} if $\mathcal{B} \subseteq \Delta_1^0(\tau')$ then $\mathcal{D} \subseteq \Delta_\eta^0(\tau')$. In both cases $\mathcal{D}' \subseteq \Pi_\eta^0(\tau')$ and $\mathcal{C} \subseteq \Delta_{\eta+1}^0(\tau')$ (by closure under complementation of \mathcal{C}), hence we are done.

Now let ν be limit and let $\langle \nu_i \mid i \in \omega \rangle$ be any increasing sequence of ordinals cofinal in ν such that $\nu_i > 1$ for every i . Since $\mathcal{C} \subseteq \Delta_{\mu+\nu}^0$ there must be some $\mathcal{D} = \{D_{n,k} \mid n, k \in \omega\} \subseteq \Delta_{<(\mu+\nu)}^0$ such that $C_n = \bigcup_{k \in \omega} D_{n,k}$. Put $\mathcal{D}_i = \{D_{n,k} \in \mathcal{D} \mid D_{n,k} \in \Delta_{\mu+\nu_i}^0\}$ for every i , so that $\mathcal{D} = \bigcup_{i \in \omega} \mathcal{D}_i$. Applying the inductive hypothesis to each \mathcal{D}_i and using AC $_\omega(\mathbb{R})$, we can find for every i a collection $\mathcal{B}_i \subseteq \Delta_{\mu+1}^0$ such that if τ' is any topology on \mathbb{R} for which $\mathcal{B}_i \subseteq \Delta_1^0(\tau')$ then $\mathcal{D}_i \subseteq \Delta_{\nu_i}^0(\tau')$. Put now $\mathcal{B} = \bigcup_{i \in \omega} \mathcal{B}_i$. Then $\mathcal{B} \subseteq \Delta_{\mu+1}^0$ and if τ' is such that $\mathcal{B} \subseteq \Delta_1^0(\tau')$ then $\mathcal{D} \subseteq \Delta_{<\nu}^0(\tau')$ and hence $\mathcal{C} \subseteq \Delta_\nu^0(\tau')$. \square

Recall now the following classical fact: if X is a zero-dimensional Polish space then there is a closed set $F \subseteq \mathbb{R}$ and an homeomorphism $H: F \rightarrow X$.

Proposition 6.2 (ZF + AC $_\omega(\mathbb{R})$). *Let $h: \mathbb{R} \rightarrow \mathbb{R}$ be in $\mathcal{B}_{\mu+\nu}$ (for some countable ordinals μ and ν). Then there are $f \in \mathcal{B}_\mu$ and $g \in \mathcal{B}_\nu$ such that $h = g \circ f$.*

Proof. Let τ be the usual topology on \mathbb{R} . If $\mu = 0$ or $\nu = 0$ the result is trivial (simply take $f = \text{id}$ and $g = h$ or, respectively, $f = h$ and $g = \text{id}$). Hence we can assume $\mu, \nu > 0$. Put $\mathcal{C} = \{h^{-1}(\mathbf{N}_s) \mid s \in {}^{<\omega}\omega\} \subseteq \Delta_{\mu+\nu+1}^0$. Let $\mathcal{B} \subseteq \Delta_{\mu+1}^0$ be obtained as in the previous lemma, that is such that for any topology τ' if $\mathcal{B} \subseteq \Delta_1^0(\tau')$ then $\mathcal{C} \subseteq \Delta_{\nu+1}^0(\tau')$. Apply Lemma 4.3 to \mathcal{B} in order to obtain a zero-dimensional Polish topology τ' such that $\mathcal{B} \subseteq \Delta_1^0(\tau')$ and let $F \subseteq \mathbb{R}$ be a closed set such that $H: (F, \tau') \rightarrow (\mathbb{R}, \tau')$ is an homeomorphism. Finally let $r: \mathbb{R} \rightarrow F$ be a retraction. Now put $g = h \circ H \circ r: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$ and $f = H^{-1}: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau)$. It is easy to check that $h: (\mathbb{R}, \tau') \rightarrow (\mathbb{R}, \tau)$ is of Baire class ν , and thus also g is of

Baire class ν . Moreover, since $H^{-1}: (\mathbb{R}, \tau') \rightarrow (F, \tau)$ is continuous and $\Delta_1^0(\tau') \subseteq \Delta_{\mu+1}^0(\tau)$, we have that f is of Baire class μ . Thus we have only to prove that $g \circ f = h$. Since $\text{range}(H^{-1}) = F$, we have that $r(H^{-1}(x)) = \text{id}(H^{-1}(x)) = H^{-1}(x)$ for every $x \in \mathbb{R}$. But then

$$g \circ f(x) = h(H(r(H^{-1}(x)))) = h(H(H^{-1}(x))) = h(x). \quad \square$$

Observe that the same statement is true if we replace h with a $\Sigma_{\mu+\nu}^0$ -measurable function (with $\mu, \nu > 1$) and we require that there are a Σ_μ^0 -measurable function and a Σ_ν^0 -measurable function whose composition gives h .

Remark 6.3. The previous proposition can be applied also to other Polish spaces \mathcal{X} (clearly we can assume again that $\mu, \nu \neq 0$, otherwise the result is trivial). In fact the same argument shows that for every $h: \mathcal{X} \rightarrow \mathcal{X}$ of Baire class $\mu + \nu$ there are $f: \mathcal{X} \rightarrow \mathbb{R}$ of Baire class μ and $g: \mathbb{R} \rightarrow \mathcal{X}$ of Baire class ν such that $h = g \circ f$. Moreover, if we assume that \mathcal{X} is (uncountable and) not \mathbf{K}_σ , the same result remains true also replacing f and g with two functions $f', g': \mathcal{X} \rightarrow \mathcal{X}$ of Baire class μ and ν , respectively. In fact in this case there is a closed set F of \mathcal{X} which is homeomorphic to \mathbb{R} via some function H' , hence one can define $f' = H'^{-1} \circ f$ and $g' = (g \circ H' \upharpoonright F) \cup (f_0 \upharpoonright \mathcal{X} \setminus F)$, where f and g are obtained as in the previous proof and f_0 is any constant function, and check that they are still of the correct Baire class. Finally, this last version of Proposition 6.2 can be further extended to *every* uncountable Polish space \mathcal{X} if we assume $\nu \neq 1$: in fact in this case we can use the fact that every zero-dimensional Polish space is homeomorphic to some \mathbf{G}_δ subspace of the Cantor space ${}^\omega 2$, which is in turn homeomorphic to a closed subset of \mathcal{X} . Therefore any zero-dimensional Polish space is homeomorphic to a \mathbf{G}_δ set G of \mathcal{X} via some function H' , and we can define f' and g' as above but replacing F with G .

Theorem 6.4 (ZF + AC $_\omega(\mathbb{R})$). *Let α be a nonzero countable ordinal. Then the closure under composition of \mathcal{B}_α is exactly $\bigcup_{\mu < \xi} \mathcal{B}_\mu$, where $\xi = \alpha \cdot \omega$ is the least additively closed ordinal above α .*

Proof. One direction is trivial (since the composition of n Baire class α functions is in $\mathcal{B}_{\alpha \cdot n}$). Suppose now that h belongs to $\mathcal{B}_{\alpha \cdot n}$ for some $1 \leq n \in \omega$. We will prove by induction on n that h belongs to the closure under composition of \mathcal{B}_α . If $n = 1$ there is nothing to prove, while if $n = m + 1$ (for some $m \geq 1$) then $h \in \mathcal{B}_{\alpha \cdot m + \alpha}$ and we can apply Proposition 6.2 to get $f \in \mathcal{B}_{\alpha \cdot m}$ and $g \in \mathcal{B}_\alpha$ such that $h = g \circ f$. Applying now the inductive hypothesis to f we get the result. \square

By the previous theorem, we are naturally led to take any countable additively closed ordinal ξ (recall that ξ is additively closed if and only if either $\xi = 0, 1$ or $\xi = \omega^\mu$ for some ordinal μ) and study the degree structure induced by

$$\mathcal{B}_\xi = \mathcal{B}_{<\xi} = \bigcup_{\mu < \xi} \mathcal{B}_\mu$$

(by the rule of composition above, \mathcal{B}_ξ is closed under composition and hence it is a Borel set of reductions). Since it is straightforward to check that \mathcal{B}_ξ is a good Borel reducibility (and therefore has the **SDP**), and that $\Delta_{\mathcal{B}_\xi} = \Delta_{<\xi}^0$ (which in particular implies that \mathcal{B}_ξ is of type III), we can apply Theorem 4.8 to get that \mathcal{B}_ξ is equivalent to any (regular) chain of reductions of rank ξ , and thus induces the same degree-structure.

This equivalence is non trivial (at least for $\xi = \omega$): in fact we will show that \mathcal{B}_ω is not contained in $\bigcup_{n \in \omega} \mathcal{D}_n$ by proving that there is a Baire class 1 function which is not in \mathcal{D}_n for any $n \in \omega$ (this discussion will also cover the missing proofs about the Pawlikowski function in [8]). First let us recall the definition of the Pawlikowski

function P from [10]. Let $\omega + 1$ have the order topology and consider the space ${}^\omega(\omega + 1)$ endowed with the corresponding product topology. It is easy to check that ${}^\omega(\omega + 1)$ is perfect, zero-dimensional and compact, hence it is homeomorphic to the Cantor space ${}^\omega 2$. Let $\gamma: \omega + 1 \rightarrow \omega$ be the bijection defined by $\gamma(\omega) = 0$ and $\gamma(n) = n + 1$ for any $n \in \omega$, and define $P: {}^\omega(\omega + 1) \rightarrow \mathbb{R}$ using γ coordinatewise, i.e. putting $P(x) = \langle \gamma(x(n)) \mid n \in \omega \rangle$. Define also (again coordinatewise) a “partial” function

$$\hat{\gamma}: {}^{<\omega}(\omega + 1) \rightarrow {}^{<\omega}\omega: s \mapsto \langle \gamma(s(i)) \mid i < \text{lh}(s) \rangle,$$

and note that both P and $\hat{\gamma}$ are bijection between the corresponding spaces.

Given $\tau \in {}^{<\omega}(\omega + 1)$, consider the set $C_\tau = \{x \in {}^\omega(\omega + 1) \mid \tau \subseteq x\}$: by simple arguments, it turns out that C_τ is always a closed set, has empty interior if and only if there is some $i < \text{lh}(\tau)$ such that $\tau(i) = \omega$, and is also open (hence a clopen set) if and only if $\tau(i) \neq \omega$ for every $i < \text{lh}(\tau)$. In particular, this implies that for every $n \in \omega$ the set $K_n = \{x \in {}^\omega(\omega + 1) \mid x(n) = \omega\}$ is a closed set with empty interior and hence a nowhere dense proper closed set (from this fact one can also derive that ${}^\omega\omega$ is a proper \mathbf{G}_δ subset of ${}^\omega(\omega + 1)$ which is also comeager and dense in it).

Lemma 6.5 (ZF + AC $_\omega(\mathbb{R})$). *Let \mathcal{X} be any zero-dimensional space. Let $\alpha < \omega_1$ be a nonzero ordinal and let $\langle \alpha_n \mid n \in \omega \rangle$ be an increasing sequence of ordinals smaller than α and cofinal in it. For every family of sets $\{P_n \subseteq {}^\omega\mathcal{X} \mid n \in \omega\}$ such that P_n is $\mathbf{\Pi}_{\alpha_n}^0$ -complete (for every $n \in \omega$), the set $S \subseteq {}^\omega\mathcal{X}$ defined by*

$$S = \{x \in {}^\omega\mathcal{X} \mid \exists n (\pi_n^\mathcal{X}(x) \in P_n)\}$$

is a $\mathbf{\Sigma}_\alpha^0$ -complete set. In particular, if $P \subseteq {}^\omega\mathcal{X}$ is a $\mathbf{\Pi}_\alpha^0$ -complete set then $S = \{x \in {}^\omega\mathcal{X} \mid \exists n (\pi_n^\mathcal{X}(x) \in P)\}$ is a $\mathbf{\Sigma}_{\alpha+1}^0$ -complete set.

Proof. Clearly $S \in \mathbf{\Sigma}_\alpha^0$. Let now Q be any set in $\mathbf{\Sigma}_\alpha^0$: by definition there is an increasing sequence $\langle \beta_k \mid k \in \omega \rangle$ of ordinals smaller than α such that $Q = \bigcup_k R_k$ for some sets $R_k \in \mathbf{\Pi}_{\beta_k}^0$. Since the sequence $\langle \alpha_n \mid n \in \omega \rangle$ is increasing and cofinal in α we can find a subsequence $\langle \alpha_{n_k} \mid k \in \omega \rangle$ such that $\beta_k \leq \alpha_{n_k}$ for any $k \in \omega$. Moreover, since every P_n is $\mathbf{\Pi}_{\alpha_n}^0$ -complete we can choose a sequence of points $\langle x_n \mid n \in \omega \rangle$ such that $x_n \notin P_n$ for every $n \in \omega$. Now define a sequence of continuous functions $\langle f_n \mid n \in \omega \rangle$ by letting f_{n_k} be any continuous reduction of R_k in P_{n_k} (which exists since $R_k \in \mathbf{\Pi}_{\beta_k}^0 \subseteq \mathbf{\Pi}_{\alpha_{n_k}}^0$), and f_n be the constant function with value x_n if there is no $k \in \omega$ such that $n = n_k$. Finally, put $f = \bigotimes_n f_n$. Clearly f is continuous and it is not hard to check that it reduces Q to S , i.e. that $x \in Q \iff f(x) \in S$ for every $x \in {}^\omega\mathcal{X}$. \square

Now we are ready to prove the following proposition which gives the exact complexity of P .

Proposition 6.6 (ZF + AC $_\omega(\mathbb{R})$). *The function P is of Baire class 1 and is in \mathbf{D}_ω but not in \mathbf{D}_n (for any nonzero $n \in \omega$).*

Proof. Since $P^{-1}(\mathbf{N}_s) = C_{\hat{\gamma}^{-1}(s)}$ for every $s \in {}^{<\omega}\omega$, we have that $P^{-1}(U)$ is the union of countably many closed sets for any open set $U \subseteq \mathbb{R}$: hence P is of Baire class 1 (this also implies that $P \in \mathbf{D}_\omega$ since every Baire class n function is in \mathbf{D}_ω). It remains only to prove that P is not in \mathbf{D}_n for any $n \in \omega$. First define

$$\begin{aligned} S_1 &= \{x \in \mathbb{R} \mid \exists n (x(n) = 0)\} \\ S_{n+1} &= \{x \in \mathbb{R} \mid \exists n (\pi_n(x) \notin S_n)\} \end{aligned}$$

for $n \geq 1$. One can inductively check that $S_n \subseteq \mathbb{R}$ is a $\mathbf{\Sigma}_n^0$ set, and that $P^{-1}(S_n) \subseteq {}^\omega(\omega + 1)$ is a complete (and hence also proper) $\mathbf{\Sigma}_{n+1}^0$ set (use Lemma 6.5 for the

inductive step). Passing to the complements, we have that $\neg S_n \in \mathbf{\Pi}_n^0 \subseteq \mathbf{\Delta}_{n+1}^0$ but $P^{-1}(\neg S_n) \notin \mathbf{\Delta}_{n+1}^0$, i.e. P is not a $\mathbf{\Delta}_{n+1}^0$ -function. \square

The function P is defined from ${}^\omega(\omega + 1)$ to \mathbb{R} , while we are interested in functions from \mathbb{R} into itself. Nevertheless it is easy to see how to obtain from P a function $\hat{P}: \mathbb{R} \rightarrow \mathbb{R}$ with the same complexity. Let $h: {}^\omega(\omega + 1) \rightarrow {}^\omega 2$ be any homeomorphism between ${}^\omega(\omega + 1)$ and the Cantor space ${}^\omega 2$ (which is a closed subspace of \mathbb{R}). Define $\hat{P}: \mathbb{R} \rightarrow \mathbb{R}$ by letting $\hat{P}(x) = P(h^{-1}(x))$ if $x \in {}^\omega 2$ and $\hat{P}(x) = \vec{0}$ otherwise. Following [10], for every $f: X_1 \rightarrow Y_1$ and $g: X_2 \rightarrow Y_2$ put $f \sqsubseteq g$ just in case there are two embeddings $\varphi: X_1 \rightarrow X_2$ and $\psi: f(X_1) \rightarrow Y_2$ such that $\psi \circ f = g \circ \varphi$. Clearly, if $f \sqsubseteq g$ and g is a $\mathbf{\Delta}_\xi^0$ -function (respectively, a Baire class ξ function) then also f is a $\mathbf{\Delta}_\xi^0$ -function (respectively a Baire class ξ function), and therefore if f is not a $\mathbf{\Delta}_\xi^0$ -function (resp. a Baire class ξ function) then neither g is a $\mathbf{\Delta}_\xi^0$ -function (resp. a Baire class ξ function). Since it is not hard to prove that \hat{P} is still a Baire class 1 function and that h and the identity function witness $P \sqsubseteq \hat{P}$, we have that $\hat{P} \in D_\omega$ but $\hat{P} \notin D_n$ for any $n \in \omega$, hence we are done.

7. COMPARING HIERARCHIES OF DEGREES

All sets of functions considered in this section are assumed to be good Borel reducibilities. Let \mathcal{G}_μ denote an arbitrary good Borel set of reductions with $\Delta_{\mathcal{G}_\mu} = \mathbf{\Delta}_\mu^0$ (in particular \mathcal{G}_μ is always of type II). To clarify the relationship between (the degree-structures induced by) different good Borel reducibilities, note that each \mathcal{F} of type I induces the finest possible hierarchy (in particular finer than the hierarchy induced by any \mathcal{G}_1 , which is in some sense the next “level of reducibility”), and each \mathcal{H} of type III with $\Delta_{\mathcal{H}} = \mathbf{\Delta}_{<\xi}^0$ (for ξ a countable limit ordinal) induces an hierarchy of degrees which is coarser than the hierarchy of the \mathcal{G}_μ -degrees (for any $\mu < \xi$), and finer than the hierarchy of the \mathcal{G}_ξ -degrees. Finally Bor , and the sets of reductions with the same characteristic set, gives the coarsest hierarchy. By part vi) of Theorem 2.2, it is clear that for an \mathcal{F} -hierarchy being coarser than the \mathcal{F}' -hierarchy amount to the fact that the \mathcal{F} -selfdual degrees are obtained gluing together many \mathcal{F}' -degrees: therefore to understand how the \mathcal{F} -structure can be obtained from the finer ones we must describe how each \mathcal{F} -selfdual degree is constructed.

The first case, that is when we want to compare the \mathcal{F} -hierarchy (for \mathcal{F} of type I) with the \mathcal{G}_1 -hierarchy, is clearly solved by the Steel–Van Wesep Theorem, which says that $A \leq_W \neg A \iff A \leq_L \neg A$: in fact since $L \subseteq \mathcal{F} \subseteq W \simeq \mathcal{G}_1$ it must be the case that $A \leq_{\mathcal{G}_1} \neg A$ if and only if $A \leq_{\mathcal{F}} \neg A$, and as $s_{\text{Lip}}(A) \leq_W A$ we get that each \mathcal{G}_1 -selfdual degree is exactly the union of a (maximal) ω_1 -chain of consecutive \mathcal{F} -selfdual degrees.

Now consider \mathcal{H} of type III as above: as $\mathcal{H} \simeq \bigcup_{\mu < \xi} \mathcal{G}_\mu$, it is clear that if $A \leq_{\mathcal{H}} \neg A$ then $[A]_{\mathcal{H}} = \bigcup_{\mu < \xi} [A]_{\mathcal{G}_\mu}$. Therefore the \mathcal{H} -hierarchy is the minimal degree-structure which is refined by all the \mathcal{G}_μ -structures.

Finally, to compare the \mathcal{H} -hierarchy with the \mathcal{G}_ξ -hierarchy, for $\vec{\mathcal{F}}$ a regular chain of reductions first define⁸ $s_\xi^\alpha[B]_{\vec{\mathcal{F}}}$ (for $B \subseteq \mathbb{R}$ and $1 \leq \alpha < \omega_1$) by letting $s_\xi^1[B]_{\vec{\mathcal{F}}} = [B]_{\vec{\mathcal{F}}}$, $s_\xi^\alpha[B]_{\vec{\mathcal{F}}} = [\bigoplus_n C_n]_{\vec{\mathcal{F}}}$ (where $C_n \in s_\xi^{\alpha_n}[B]_{\vec{\mathcal{F}}}$ and the α_n ’s are increasing and cofinal in α) if α is limit, and $s_\xi^\alpha[B]_{\vec{\mathcal{F}}} = [s_{\vec{\mathcal{F}}}(C)]_{\vec{\mathcal{F}}}$ (where $C \in s_\xi^{\alpha'}[B]_{\vec{\mathcal{F}}}$) if $\alpha = \alpha' + 1$. (Note that if $B \leq_{\vec{\mathcal{F}}} \neg B$ then the $s_\xi^\alpha[B]_{\vec{\mathcal{F}}}$ ’s are exactly the ω_1 -chain of consecutive $\vec{\mathcal{F}}$ -selfdual degrees which follows $[B]_{\vec{\mathcal{F}}}$.) Moreover, given a pointclass Γ and a nonzero

⁸We must define the ω_1 -chain on the $\vec{\mathcal{F}}$ -degrees (rather than on sets) because the Perfect Set Property, which already follows from SLO^L , forbids the possibility of having an ω_1 -chain of sets of bounded Borel rank.

ordinal $\mu < \omega_1$, define

$$\text{PU}_\mu(\Gamma) = \left\{ \bigcup_n (A_n \cap D_n) \mid A_n \in \Gamma \text{ and } \langle D_n \mid n \in \omega \rangle \text{ is a } \Delta_\mu^0\text{-partition of } \mathbb{R} \right\}$$

and

$$\text{SU}_\mu(\Gamma) = \left\{ \bigcup_n (A_n \cap D_n) \mid A_n \in \Gamma, D_n \in \Delta_\mu^0 \text{ and } D_n \cap D_m = \emptyset \text{ if } n \neq m \right\}.$$

Finally, for μ limit and μ_n 's (strictly) increasing and cofinal in μ define $\text{SU}_{<\mu,\alpha}(\Gamma)$ by the following induction on $\alpha < \omega_1$ (note the definition is independent from the choice of the μ_n 's):

$$\text{SU}_{<\mu,\alpha}(\Gamma) = \begin{cases} \Gamma & \text{if } \alpha = 0 \\ \bigcup_n \text{SU}_{\mu_n}(\bigcup_{\alpha' < \alpha} \text{SU}_{<\mu,\alpha'}(\Gamma)) & \text{if } \alpha > 0. \end{cases}$$

Proposition 7.1 (ZF + AC $_\omega(\mathbb{R})$). *Let $\vec{\mathcal{F}}$ be a regular chain of reductions of rank ξ (for ξ a countable limit ordinal). For $A, B \subseteq \mathbb{R}$, $A \leq_{D_\xi^W} B$ if and only if $A \leq_W C$ for some $C \in s_\xi^\alpha[B]_{\vec{\mathcal{F}}}$ and $\alpha < \omega_1$.*

Proof. Let $\mathbf{\Gamma} = \mathbf{\Gamma}(B) = \{D \subseteq \mathbb{R} \mid D \leq_W B\}$ be the boldface pointclass generated by B . It is immediate to check that $A \leq_{D_\xi^W} B \iff A \in \text{PU}_\xi(\mathbf{\Gamma})$. By Theorem E.4 of chapter IV of Wadge's [13], $\text{PU}_\xi(\mathbf{\Gamma}) = \bigcup_{\alpha < \omega_1} \text{SU}_{<\xi,\alpha}(\mathbf{\Gamma})$, so let α be smallest such that $A \in \text{SU}_{<\xi,\alpha}(\mathbf{\Gamma})$. We will prove by induction on α that $A \leq_W C$ for some $C \in s_\xi^{\alpha+1}[B]_{\vec{\mathcal{F}}}$. If $\alpha = 0$, then $A \in \text{SU}_{<\xi,0}(\mathbf{\Gamma}) = \mathbf{\Gamma}$ and therefore $A \leq_W B$ (by definition of $\mathbf{\Gamma}$) and obviously $B \in [B]_{\vec{\mathcal{F}}} = s_\xi^1[B]_{\vec{\mathcal{F}}}$. Now assume $\alpha > 0$, and let n be such that $A \in \text{SU}_{\mu_n}(\bigcup_{\alpha' < \alpha} \text{SU}_{<\mu,\alpha'}(\mathbf{\Gamma}))$, so that $A = \bigcup_m (A_m \cap D_m)$ where $D_m \in \Delta_{\mu_n}^0$, $D_m \cap D_{m'} = \emptyset$ if $m \neq m'$, and $A_m \in \text{SU}_{<\mu,\alpha_m}(\mathbf{\Gamma})$ for some $\alpha_m < \alpha$ (depending on m). By inductive hypothesis, $A_m \leq_W C_m$ for some $C_m \in s_\xi^{\alpha_m+1}[B]_{\vec{\mathcal{F}}}$, and therefore $A_m \leq_W \bigoplus_m C_m$ for every m . Moreover $\bigoplus_m C_m \in s_\xi^\alpha[B]_{\vec{\mathcal{F}}}$ as $\alpha = \sup\{\alpha_m + 1 \mid m \in \omega\}$ by its minimality.

Claim 7.1.1. $A \leq_W \Sigma^{\mu_n+1}(\bigoplus_m C_m)$.

Proof of Claim. Let $\langle \mu_n \mid n \in \omega \rangle$ be the type of $\vec{\mathcal{F}}$, P be the complete $\Pi_{\mu_n}^0$ -set used to define the operator Σ^{μ_n+1} , and f_m be continuous functions such that f_0 reduces $\neg \bigcup_m D_m$ to P , $f_{2(m+1)}$ reduces D_m to P , f_1 is constant with value $y \notin \bigoplus_m C_m$, and f_{2m+3} is a reduction of A_m to $\bigoplus_m C_m$: it is easy to check that $\bigotimes_m f_m$ reduces A to $\Sigma^{\mu_n+1}(\bigoplus_m C_m)$ as required. \square *Claim*

Since $\mu_n < \mu_n + 1 \leq \mu_{n+1}$, we get $\Sigma^{\mu_n+1}(\bigoplus_m C_m) \leq_W \Sigma^{\mu_{n+1}}(\bigoplus_m C_m) \leq_W s_{\vec{\mathcal{F}}}(\bigoplus_m C_m) \in s_\xi^{\alpha+1}[B]_{\vec{\mathcal{F}}}$ and hence we are done. \square

As a corollary of Proposition 7.1 one gets a Steel–Van Wesep-style theorem for higher levels.

Theorem 7.2. *Let \mathcal{G}_ξ be as above and $\vec{\mathcal{F}}$ be a regular chain of reductions of rank ξ . Then $A \leq_{\mathcal{G}_\xi} \neg A$ if and only if $A \leq_{\vec{\mathcal{F}}} \neg A$. In particular, $A \leq_{D_\xi} \neg A$ implies that $A \leq_{D_\mu} \neg A$ for some $\mu < \xi$.*

Proof. One direction is easy, as $\bigcup_{\mu < \xi} D_\mu \subseteq D_\xi \simeq \mathcal{G}_\xi$. For the other direction let B be L-minimal in $[A]_{\mathcal{G}_\xi}$, so that $B \leq_L \neg B$. As $\mathcal{G}_\xi \simeq D_\xi^W$, apply Proposition 7.1 and let α be minimal such that $A \leq_W C$ for some $C \in s_\xi^\alpha[B]_{\vec{\mathcal{F}}}$. Since $A <_{\vec{\mathcal{F}}} C$ contradicts the minimality of α , we must have $C \leq_{\vec{\mathcal{F}}} A$ and therefore $A \equiv_{\vec{\mathcal{F}}} C$: but as C is $\vec{\mathcal{F}}$ -selfdual (since B is) we get $A \leq_{\vec{\mathcal{F}}} \neg A$ as desired. \square

All this discussion solves the problem of comparing the \mathcal{H} -hierarchy with the \mathcal{G}_ξ -hierarchy: using the fact that $\mathcal{G}_\xi \simeq \mathcal{D}_\xi^W$, $\mathcal{H} \simeq \vec{\mathcal{F}}$ (where $\vec{\mathcal{F}}$ is any regular chain of rank ξ), and $s_{\vec{\mathcal{F}}}(A) \leq_{\mathcal{D}_\xi^W} A$, we get that each \mathcal{G}_ξ -selfdual degree is exactly the union of a (maximal) ω_1 -chain of consecutive \mathcal{H} -selfdual degrees.

APPENDIX: SOME ALTERNATIVE PROOFS OF THE **SDP**

In many of the concrete examples, one can directly prove (in a simpler way) that a Borel set of reductions \mathcal{F} has the **SDP** using the fact that $\text{Lip}(2) \subseteq \mathcal{F}$ and applying Proposition 4.2. For instance, if $\mathcal{F} = \text{Lip}$ or $\mathcal{F} = \text{UCont}$, given a set $A \leq_{\mathcal{F}} \neg A$ which is L -minimal in its \mathcal{F} -degree we can use the fact that $A \leq_W \neg A$ (since $\text{Lip} \subseteq \text{UCont} \subseteq W$) and then apply the Steel–Van Wesep Theorem to get $A \leq_L \neg A$.

In the case of a regular chains of reductions $\vec{\mathcal{F}} = \langle \mathcal{F}_n \mid n \in \omega \rangle$, we can prove that if $A \leq_{\vec{\mathcal{F}}} \neg A$ and B is L -minimal in $[A]_{\vec{\mathcal{F}}}$ then $B \leq_L \neg B$ as follows: let n_0 be minimal such that $A \leq_{\mathcal{F}_{n_0}} \neg A$, so that $A \leq_{\mathcal{F}_m} \neg A$ for every $m \geq n_0$. Moreover, for every $m \geq n_0$ let B_m be L -minimal in $[A]_{\mathcal{F}_m}$ (so that $B_m \leq_L \neg B_m$ by Corollary 5.4 in [8]), and note that if $0 \leq k \leq m$ then $B_m \leq_L C$ for every $C \in [A]_{\mathcal{F}_k}$ because $\vec{\mathcal{F}}$ is regular (in particular, if $n_0 \leq k \leq m$ then $B_m \leq_L B_k$). Since \leq_L is well-founded, there must be some $n_1 \geq n_0$ such that $B_m \equiv_L B_{n_1}$ for every $m \geq n_1$. Put $B' = B_{n_1}$: then $B' \leq_L \neg B'$ and B' is easily seen to be L -minimal in $[A]_{\vec{\mathcal{F}}}$, so that $B' \equiv_L B$ and we are done.

However, Baire reductions \mathcal{B}_ξ are perhaps the most interesting case.

Lemma (ZF + AC $_\omega(\mathbb{R})$). *Let $A, B \subseteq \mathbb{R}$ be such that $A \leq_{\mathcal{B}_\xi} B$. Then there is some $C \equiv_{\mathcal{B}_\xi} A$ such that $C \leq_L A$ and $C \leq_L B$.*

Proof. It is enough to prove that there is some $A' \equiv_{\mathcal{B}_\xi} A$ such that $A' \leq_W A$ and $A' \leq_W B$: then applying twice Lemma 19 of [1] we get the desired C as in the proof of Lemma 8 in [4]. Let $\mu < \xi$ and $f \in \mathcal{B}_\mu$ be such that $A = f^{-1}(B)$. Applying Lemma 4.3 to the family $\{f^{-1}(\mathbf{N}_s) \mid s \in {}^{<\omega}\omega\} \subseteq \Delta_{\mu+1}^0$, we get a new zero-dimensional Polish topology $\tau' \supseteq \tau$ on \mathbb{R} such that $f: (\mathbb{R}, \tau') \rightarrow (\mathbb{R}, \tau)$ is continuous and $\Sigma_1^0(\tau') \subseteq \Sigma_{\mu+1}^0(\tau)$. Let $H: (F, \tau) \rightarrow (\mathbb{R}, \tau')$ be an homeomorphism between a closed set $F \subseteq \mathbb{R}$ and \mathbb{R} endowed with the new topology, and let $r: \mathbb{R} \rightarrow F$ be a retraction on F . Finally, put $A' = (H \circ r)^{-1}(A)$: since $H^{-1}: (\mathbb{R}, \tau) \rightarrow (\mathbb{R}, \tau')$ is of Baire class μ we get $A \leq_{\mathcal{B}_\xi} A'$, and since $H \circ r$ and $f \circ H \circ r$ are clearly continuous function from (\mathbb{R}, τ) to (\mathbb{R}, τ') which witness $A' \leq_W A$ and $A' \leq_W B$, respectively, A' is as required. \square

Let now B be L -minimal in $[A]_{\mathcal{B}_\xi}$, where $A \leq_{\mathcal{B}_\xi} \neg A$. Since $B \leq_{\mathcal{B}_\xi} \neg B$, we can apply the previous lemma to get $C \equiv_{\mathcal{B}_\xi} B$ such that $C \leq_L B$ and $C \leq_L \neg B$. By minimality of B , we must have $B \equiv_L C$ and hence $B \leq_L \neg B$.

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